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by

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Taxis-driven persistent localization in a degenerate Keller-Segel system

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Abstract

The degenerate Keller-Segel type system

$$\begin{cases} u_t = \nabla\cdot(u^{m-1}\nabla u) - \nabla\cdot(u\nabla v), & x\in\Omega,\ t>0,\\ 0 = \Delta v - \mu + u, \qquad \int_\Omega v = 0, \quad \mu = \frac{1}{|\Omega|}\int_\Omega u, & x\in\Omega,\ t>0, \end{cases}$$

is considerd in balls $\Omega = B_R(0)\subset\mathbb{R}^n$ with $n\ge 1$, $R>0$ and $m>1$.

Our main results reveal that throughout the entire degeneracy range $m\in(1,\infty)$, the interplay between degenerate diffusion and cross-diffusive attraction herein can enforce persistent localization of solutions inside a compact subset of Ω, no matter whether solutions remain bounded or blow up. More precisely, it is shown that for arbitrary $\mu>0, \sigma\in(0,1)$ and $\theta\in(0,\sigma)$ one can find $R_\star = R_\star(n,m,\mu,\sigma,\theta)>0$ such that if $R\ge R_\star$ and $u_0\in L^\infty(\Omega)$ is nonnegative and radially symmetric with $\frac{1}{|\Omega|}\int_\Omega u_0 = \mu$ and

$$\frac{1}{|B_r(0)|}\int_{B_r(0)} u_0 \ge \frac{\mu}{\theta^n} \qquad \text{for all } r\in(0,\theta R),$$

then a corresponding zero-flux type initial-boundary value problem admits a radial weak solution (u,v), extensible up to a maximal time $T_{max}\in(0,\infty]$ and satisfying $\lim_{t\nearrow T_{max}}\|u(\cdot,t)\|_{L^\infty(\Omega)}=\infty$ if $T_{max}<\infty$, which has the additional property that

$$\operatorname{supp} u(\cdot,t)\subset \overline{B}_{\sigma R}(0) \qquad \text{for all } t\in(0,T_{max}).$$

In particular, this conclusion is seen to be valid whenever u_0 is radially nonincreasing with $\operatorname{supp} u_0 \subset \overline{B}_{\theta R}(0)$.

Key words: chemotaxis; degenerate diffusion; compact support
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1 Introduction

The literature concerned with the detection of taxis-driven phenomena in Keller-Segel type boundary value problems has concentrated to a large percentage on questions related to the emergence of structures. Indeed, strong indications for the ability of such systems to describe pattern formation have become manifest not only in results on richly structured equilibrium sets ([44], [37], [17], [16]), but beyond this also in a meanwhile considerable collection of findings on spontaneous singularity formation and their related mechanisms and scalings ([27], [23], [22], [41], [4], [49]; see also the surveys [24], [25] and [36]).

In comparison to this, aspects of mass propagation appear to have been understood to a significantly lower extent, and large parts of the literature in this regard seem devoted to issues naturally requiring unboundedness of the underlying physical domain as a prerequisite. In fact, various facets of wave-like transport mechanisms, mainly in contexts of particular solutions reflecting traveling fronts, have been addressed quite thoroughly in the literature over the past few years (see [43], [39], [10], [26], [31] for a small selection of examples and [48] for an overview), and in certain Cauchy problems also some statements on asymptotic self-similarity of solutions emanating from fairly general spatially decaying initial data have been derived ([38], [34]). The knowledge on possible influences of chemotactic cross-diffusion on population distributions initially confined to a bounded region, however, so far seems essentially limited to results on temporally local features such as finite speed of propagation, asserting finite speed of support propagation ([21], [32], [45]). One exception can be found in [32], where a statement on persistent localization in a Cauchy problem in $\mathbb{R}^n$ has been derived, but possibly involving large eventual positivity sets.

Main results: Quantitative control of localization in chemotaxis systems with arbitrary porous medium diffusion. In connection to the latter, the present study will be devoted to the discovery of a genuinely taxis-driven effect on spatial localization throughout evolution, arbitrarily strong in the sense that the maximum possible support can a priori be asserted to remain close to the corresponding initial positivity set.

The specific framework within which this will be examined is the zero-flux type initial-boundary value problem

$$\begin{cases} u_t = \nabla \cdot (u^{m-1}\nabla u) - \nabla \cdot (u\nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu + u, \qquad \int_\Omega v = 0, \quad \mu = \frac{1}{|\Omega|}\int_\Omega u_0, & x \in \Omega,\ t > 0, \\ (u^{m-1}\nabla u - u\nabla v)\cdot\nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \qquad (1.1)$$

in a smoothly bounded domain $\Omega = B_R(0) \subset \mathbb{R}^n$. Our subsequently standing assumption $m > 1$ will let us exclusively focus on a degenerate version of the classical Keller-Segel system ([30]), simplified here to a parabolic-elliptic variant according to a standard model reduction argument based on a fast signal diffusion assumption (see [27], [4] and the discussion at the end of this paper). As is well-known from a comprehensive literature on porous medium problems, one of the core characteristics of the taxis-free counterpart of (1.1), as formed by the corresponding problem for the scalar equation $u_t = \nabla\cdot(u^{m-1}\nabla u)$, consists in the fact that the support of any nontrivial nonnegative initial data will propagate at an at most finite speed, and coincide with all of $\overline{\Omega}$ after some finite time ([1]).

Our main results will reveal that this key spreading tendency of nonlinear diffusion can be counteracted by the attractive cross-diffusion/taxis mechanism in (1.1) in such a way that even persistent localization within the interior of Ω can be observed, throughout the entire degeneracy range $m \in (1,\infty)$.

Indeed, resorting to radial symmetry so as to make (1.1) accessible to a comparison argument to be described in Sections 2-4, we shall see that solutions to (1.1) can persistently have their spatial positivity set remain in an arbitrarily close neighborhood of their initial support, in the sense quantitatively described in the following main statement of this note:

Theorem 1.1 *Let $n \ge 1, m > 1, \mu > 0$ and $0 < \theta < \sigma < 1$. Then there exists $R_\star = R_\star(n,m,\mu,\sigma,\theta) > 0$ such that whenever $\Omega = B_R(0) \subset \mathbb{R}^n$ with some $R \ge R_\star$ and $u_0 \in L^\infty(\Omega)$ is nonnegative and radially symmetric and satisfies $\frac{1}{|\Omega|}\int_\Omega u_0 = \mu$ as well as*

$$\frac{1}{|B_r(0)|}\int_{B_r(0)} u_0 \ge \frac{\mu}{\theta^n} \qquad \text{for all } r \in (0,\theta R), \tag{1.2}$$

one can find $T_{max} \in (0,\infty]$ and at least one radial weak solution (u,v) of (1.1) in $\Omega \times (0,T_{max})$ in the sense of Definition 2.1, having the properties that

$$\text{if } T_{max} < \infty, \quad \text{then} \quad \|u(\cdot,t\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \nearrow T_{max}, \tag{1.3}$$

and that additionally

$$u(\cdot,t) \equiv 0 \quad \text{a.e. in } \Omega \setminus B_{\sigma R}(0) \qquad \text{for all } t \in (0,T_{max}). \tag{1.4}$$

A straightfoward consequence thereof replaces (1.2) with a more convenient though slightly stronger set of assumptions on the initial data:

Corollary 1.2 *Let $n \ge 1, m > 1, \mu > 0, \sigma \in (0,1)$ and $\theta \in (0,\sigma)$, and let $\Omega = B_R(0) \subset \mathbb{R}^n$ for some $R \ge R_\star$ with $R_\star = R_\star(n,m,\mu,\sigma,\theta) > 0$ as given by Theorem 1.1. Then for any nonnegative radially symmetric $u_0 \in L^\infty(\Omega)$ with $\frac{1}{|\Omega|}\int_\Omega u_0 = \mu$ which is such that u_0 is nonincreasing with respect to $|x|$, and which moreover satisfies*

$$\operatorname{supp} u_0 \subset \overline{B}_{\theta R}(0), \tag{1.5}$$

the problem (1.1) possesses a radial weak solution (u,v), extended up to a maximal time $T_{max} \in (0,\infty]$ fulfilling (1.3), which furthermore has the property that

$$\operatorname{supp} u(\cdot,t) \subset \overline{B}_{\sigma R}(0) \qquad \text{for all } t \in (0,T_{max}). \tag{1.6}$$

Remark. i) Especially in view of the circumstance that the localization result in [32] seems to strongly depend on the assumption $m > 2-\frac{2}{n}$ made there, let us emphasize that by covering the whole degeneracy range $m \in (1,\infty)$, our results particularly apply to the case when $n \ge 3$ and $m \in (1, 2-\frac{2}{n})$, in which some radial solutions to (1.1) may blow up in finite time (see [15]).

ii) As will become clear in the course of our analysis, the results of Theorem 1.1 and Corollary 1.2 extend to $\Delta D(u)$ in place of $\nabla(u^{m-1}\nabla u)$ if D is suitably regular and behaves in a way appropriately controllable by that of $u \mapsto u^m$. Further, variants of (1.1) containing arbitrary positive factors for its summands are included in our setting; cf. also the discussion near the end of this manuscript.

iii) Unlike in the scalar porous medium equation, uniqueness seems unclear unless additional assumptions on Hölder continuity are imposed ([33]). According to the standard parabolic regularity result from [42], this additional property is satisfied when $m < 3$; for stronger degeneracies, however, this seems unknown.

iv) Connections to results on so-called aggregation equations will be given in the discussion at the end of this paper.

2 Preliminaries. Preparations for a comparison argument

In order to outline the basis of our strategy, we recall a procedure well-known from [27], and prepare a reduction to a scalar parabolic setting accessible to comparison arguments: Given a nonnegative radial function $u_0 \in L^\infty(\Omega)$ in $\Omega = B_R(0) \subset \mathbb{R}^n$ with $\frac{1}{|\Omega|}\int_\Omega u_0 = \mu$, and assuming (u,v) to be a suitably regular radially symmetric solution of (1.1) in $\Omega \times (0,T)$ for some $T \in (0,\infty]$ with $u \geq 0$ a.e. in $\Omega \times (0,T)$, we observe that defining

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho,t) d\rho, \qquad s \in [0,R^n],\ t \in (0,T), \tag{2.1}$$

and, accordingly,

$$w_0(s) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u_0(\rho) d\rho, \qquad s \in [0,R^n], \tag{2.2}$$

introduces a function w which satisfies $w_s \geq 0$ a.e. in $(0,R^n) \times (0,T)$, and which should solve

$$\begin{cases} w_t = n^2 s^{2-\frac{2}{n}} w_s^{m-1} w_{ss} + nww_s - \mu s w_s, & s \in (0,R^n),\ t \in (0,T), \\ w(0,t) = 0, w(R^n,t) = \frac{\mu R^n}{n}, & t \in (0,T), \\ w(s,0) = w_0(s), & s \in (0,R^n), \end{cases} \tag{2.3}$$

in an appropriate sense (cf. also [15]). Our approach will now be guided by the idea that if from whatever source we can find *stationary subsolutions* $\underline{w} = \underline{w}(s)$ to this problem satisfying $\underline{w} \equiv \frac{\mu R^n}{n}$ on (s_0, R^n) with some $s_0 \in (0,R^n)$, then whenever u_0 is such that the function w_0 in (2.2) satisfies $w_0 \geq \underline{w}$, a comparison argument should assert that throughout evolution we have $w \geq \underline{w}$ and hence, by conservation of mass and (2.1), $\operatorname{supp} u \subset \overline{B}_{s_0^{\frac{1}{n}}}(0)$.

In order to substantiate this in the context of appropriately regularized variants of (1.1) for which a comparison principle can rigorously be derived, let us consider the non-degenerate approximations given by

$$\begin{cases} \partial_t u_\varepsilon = \nabla \cdot ((u_\varepsilon + \varepsilon)^{m-1} \nabla u_\varepsilon) - \nabla \cdot (u_\varepsilon \nabla v_\varepsilon), & x \in \Omega,\ t > 0, \\ 0 = \Delta v_\varepsilon - \mu + u_\varepsilon, \qquad \int_\Omega v_\varepsilon = 0, \quad \mu = \frac{1}{|\Omega|}\int_\Omega u_0, & x \in \Omega,\ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.4}$$

for $\varepsilon \in (0,1)$. For the family of these uniformly parabolic problems, the following quantitative version of a local-in-time existence statement can essentially be imported from known literature.

Lemma 2.1 *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ with $n \geq 1$ and $R > 0$, and let $m > 1$ and $\mu > 0$. Then for all $M > 0$ there exists a time $T(M) > 0$ with the following property: Whenever $u_0 \in L^\infty(\Omega)$ is radially symmetric with $0 \leq u_0 \leq M$ a.e. in Ω, for each $\varepsilon \in (0,1)$ the boundary value problem in (2.4) possesses at least one classical solution $(u_\varepsilon, v_\varepsilon)$ with*

$$\begin{cases} u_\varepsilon \in C^{2,1}(\overline{\Omega} \times (0, T(M))) \qquad \text{and} \\ v_\varepsilon \in C^{2,0}(\overline{\Omega} \times (0, T(M))), \end{cases} \tag{2.5}$$

for which $u_\varepsilon(\cdot, t)$ and $v_\varepsilon(\cdot, t)$ are radially symmetric for all $t \in (0, T(M))$ with

$$0 < u_\varepsilon \leq M + 1 \qquad \text{in } \Omega \times (0, T(M)), \tag{2.6}$$

and for which in addition $u_\varepsilon \in C^0([0, T(M)); L^1(\Omega))$.
Moreover, one can find $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0,1)$ and a radial weak solution (u, v) of (1.1) in $\Omega \times (0, T(M))$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and

$$u_\varepsilon \to u \quad \text{a.e. in } \Omega \times (0, T(M)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Proof. Using that for fixed $\varepsilon \in (0,1)$ the problem (2.4) is non-degenerate, one can derive all statements by combining arguments well-established in the context of parabolic-elliptic chemotaxis systems (cf. e.g. [15], [19]) with standard arguments from elliptic and parabolic regularity theory, as well as the strong maximum principle. □

In their respective versions accordingly transformed in the style of (2.1)-(2.3), these non-degenerate problems (2.4) now indeed allow for a comparison principle. The following lemma in this regard can be viewed as reducing a more general statement presented in [3, Lemma 5.1] to the particular nonlinearities present in (2.4). Although in its original formulation the corresponding statement in the latter reference requires slightly stronger regularity assumptions, a verbatim copy of its proof can readily be seen to cover also the present setting.

Lemma 2.2 *Let $L > 0$ and $T > 0$, and suppose that $\underline{w}$ and $\overline{w}$ are two functions from $C^0([0, L] \times [0, T)) \cap C^1((0, L) \times (0, T))$ for which $\underline{w}_s$ and $\overline{w}_s$ belong to $L^\infty_{loc}([0, L] \times [0, T))$, and which satisfy*

$$\underline{w}_s(s, t) > 0 \quad \text{and} \quad \overline{w}_s(s, t) > 0 \qquad \text{for all } s \in (0, L) \text{ and } t \in (0, T)$$

as well as

$$\underline{w}(\cdot, t) \in W^{2,\infty}_{loc}((0, L)) \quad \text{and} \quad \overline{w}(\cdot, t) \in W^{2,\infty}_{loc}((0, L)) \qquad \text{for all } t \in (0, T).$$

Then if for all $t \in (0, T)$ and a.e. $s \in (0, L)$,

$$\underline{w}_t - n^2 s^{2-\frac{2}{n}} (\underline{w}_s + \varepsilon)^{m-1} \underline{w}_{ss} - n\underline{w}\underline{w}_s + \mu s \underline{w}_s \leq 0 \leq \overline{w}_t - n^2 s^{2-\frac{2}{n}} (\overline{w}_s + \varepsilon)^{m-1} \overline{w}_{ss} - n\overline{w}\overline{w}_s + \mu s \overline{w}_s$$

if

$$\underline{w}(s, 0) \leq \overline{w}(s, 0) \qquad \text{for all } s \in (0, L)$$

and if

$$\underline{w}(0, t) \leq \overline{w}(0, t) \quad \text{and} \quad \underline{w}(L, t) \leq \overline{w}(L, t) \qquad \text{for all } t \in (0, T),$$

we have

$$\underline{w}(s, t) \leq \overline{w}(s, t) \qquad \text{for all } s \in [0, L] \text{ and } t \in [0, T).$$

In order to formulate a template for our final conclusion thereof concerning the original problem (1.1), let us now specify the concept of weak solvability to be pursued in the sequel.

Definition 2.1 *Let $n \geq 1, R > 0, m > 0, \mu > 0$ and $T \in (0, \infty]$, and assume that $u_0 \in L^\infty(\Omega)$ is nonnegative and radially symmetric in $\Omega = B_R(0) \subset \mathbb{R}^n$ with $\frac{1}{|\Omega|}\int_\Omega u_0 = \mu$. Then by a* radial weak solution *of (1.1) in $\Omega \times (0,T)$ we mean a pair of radially symmetric functions u and v on $\Omega \times (0,\infty)$ such that $u \geq 0$ a.e. in $\Omega \times (0,\infty)$, that*

$$\begin{cases} u \in L^\infty_{loc}\left(\overline{\Omega} \times [0,T)\right) \quad \text{with} \quad u^m \in L^2_{loc}\left([0,T); W^{1,2}(\Omega)\right) \qquad \text{and} \\ v \in L^\infty_{loc}\left([0,T); W^{1,2}(\Omega)\right) \end{cases} \tag{2.7}$$

and that both

$$-\int_0^T \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = -\frac{1}{m}\int_0^T \int_\Omega \nabla u^m \cdot \nabla\varphi + \int_0^T \int_\Omega u\nabla v \cdot \nabla\varphi \tag{2.8}$$

and

$$\int_0^T \int_\Omega \nabla v \cdot \nabla\varphi = -\mu\int_0^T \int_\Omega \varphi + \int_0^T \int_\Omega u\varphi \tag{2.9}$$

hold for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,T))$.

Remark. i) As (2.7) asserts that for any radial weak solution (u,v) in $\Omega \times (0,T)$ we know that $-\frac{1}{m}\nabla u^m + u\nabla v \in L^2_{loc}(\overline{\Omega} \times [0,T))$ and that hence $u_t \in L^2_{loc}\left([0,T); (W^{1,2}(\Omega))^\star\right)$, it follows that after redefining u on a null set of times we may assume that actually u belongs to $C^0([0,T); L^2(\Omega))$.

ii) On the basis of (2.8) it can readily be checked that according to i), any radial weak solution (u,v) in $\Omega \times (0,T)$ enjoys the mass conservation property

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \qquad \text{for all } t \in (0,T). \tag{2.10}$$

The following lemma now combines Lemma 2.1 with Lemma 2.2 to establish the main result of this section, which fleshes out the essence of our subsequent ambitions.

Lemma 2.3 *Let $n \geq 1$, $R > 0$, $m > 1$ and $\mu > 0$, and suppose that there exists a family of functions $\mathcal{F} \subset W^{2,\infty}((0,R^n))$ such that for all $\underline{w} \in \mathcal{F}$ we have $\underline{w}_s > 0$ in $[0,R^n]$, and that for any such $\underline{w}$ we can find $\varepsilon_\star(\underline{w}) \in (0,1)$ with the property that*

$$n^2 s^{2-\frac{2}{n}}(\underline{w}_s + \varepsilon)^{m-1}\underline{w}_{ss} + n\underline{w}\,\underline{w}_s - \mu s\underline{w}_s \geq 0 \qquad \text{for a.e. } s \in (0,R^n) \text{ and all } \varepsilon \in (0,\varepsilon_\star(\underline{w})). \tag{2.11}$$

Moreover, suppose that $u_0 \in L^\infty(\Omega)$ is nonnegative and radially symmetric with $\frac{1}{|\Omega|}\int_\Omega u_0 = \mu$, and that the function w_0 from (2.2) satisfies

$$w_0(s) \geq \underline{w}(s) \qquad \text{for all } s \in (0,R^n) \text{ and each } \underline{w} \in \mathcal{F}. \tag{2.12}$$

Then there exist $T > 0$ and a radial weak solution (u,v) of (1.1) in $\Omega \times (0,T)$ for which in addition we have

$$w(s,t) \geq \underline{w}(s) \qquad \text{for all } s \in (0,R^n),\ t \in (0,T) \text{ and } \underline{w} \in \mathcal{F}, \tag{2.13}$$

with w as defined in (2.1).

Proof. An application of Lemma 2.1 to $M := \|u_0\|_{L^\infty(\Omega)}$ yields $T(M) > 0$ such that for all $\varepsilon \in (0,1)$, the boundary value problem in (2.4) possesses a radially symmetric classical solution $(u_\varepsilon, v_\varepsilon) \in C^{2,1}(\overline{\Omega} \times (0,T(M))) \times C^{2,0}(\overline{\Omega} \times (0,T(M)))$ with $0 \le u_\varepsilon \le M+1$ and the regularity properties listed in (2.5). Since in particular u_ε lies in $C^0([0,T(M)); L^1(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T(M)))$, it follows that $w_\varepsilon(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u_\varepsilon(\rho,t)d\rho$, $s \in [0,R^n]$, $t \in [0,T(M))$, defines a function w_ε on $[0,R^n] \times [0,T(M))$ which does not only belong to $C^{2,1}((0,R^n] \times (0,T(M)))$ but moreover is continuous in all of $[0,R^n] \times [0,T(M))$, and which, as can be seen similarly to the derivation of (2.3), is a classical solution of the problem

$$\begin{cases} \partial_t w_\varepsilon = n^2 s^{2-\frac{2}{n}} (\partial_s w_\varepsilon + \varepsilon)^{m-1} \partial_s^2 w_\varepsilon + n w_\varepsilon \partial_s w_\varepsilon - \mu s \partial_s w_\varepsilon, & s \in (0,R^n),\ t \in (0,T(M)), \\ w_\varepsilon(0,t) = 0, w_\varepsilon(R^n,t) = \frac{\mu R^n}{n}, & t \in (0,T(M)), \\ w_\varepsilon(s,0) = w_0(s), & s \in (0,R^n). \end{cases} \tag{2.14}$$

Now for fixed $\underline{w} \in \mathcal{F}$ and $\varepsilon_\star = \varepsilon_\star(\underline{w})$ taken from our hypothesis, (2.14) together with (2.12) clearly implies that $w_\varepsilon(0,t) \ge \underline{w}(0)$ and $w_\varepsilon(R^n,t) \ge \underline{w}(R^n)$ for all $t \in (0,T(M))$ and any $\varepsilon \in (0,\varepsilon_\star)$, whence in view of the subsolution feature in (2.11) the comparison principle from Lemma 2.2 becomes applicable so as to assert that the initial ordering property (2.12) is inherited by w_ε in the sense that

$$w_\varepsilon(s,t) \ge \underline{w}(s) \qquad \text{for all } s \in (0,R^n), \text{ any } t \in (0,T(M)) \text{ and each } \varepsilon \in (0,\varepsilon_\star). \tag{2.15}$$

Now since from Lemma 2.1 we moreover know that with $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ as provided there and some radial weak solution (u,v) of (1.1) in $\Omega \times (0,T(M))$ we have $u_\varepsilon \to u$ a.e. in $\Omega \times (0,T(M))$ and thus, by boundedness of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty(\Omega \times (0,T(M)))$, also $u_\varepsilon(\cdot,t) \to u(\cdot,t)$ in $L^1(\Omega)$ for a.e. $t \in (0,T(M))$ as $\varepsilon = \varepsilon_j \searrow 0$, according to the Fubini-Tonelli theorem and the dominated convergence theorem. Therefore, (2.15) implies that for this solution and the correspondingly defined function w from (2.1) we have

$$w(\cdot,t) \ge \underline{w} \qquad \text{in } (0,R^n) \text{ for a.e. } t \in (0,T(M)) \text{ and all } \underline{w} \in \mathcal{F}.$$

As also w is continuous in $\overline{\Omega} \times [0,T(M))$ by $L^1(\Omega)$-valued continuity of u, this establishes (2.13). □

3 Construction of stationary subsolutions

In accordance with the above, the purpose of this core section now consists in the constrution of an appropriate family $\mathcal{F}$ of functions to be used in Lemma 2.3. Our candidates for such stationary subsolutions will exhibit a tripartite structure, the outer two parts of which will be described in the following.

Lemma 3.1 *Let $n \ge 1, R > 0, m > 1, \mu > 0, 0 < \lambda < \kappa < 1, \gamma \ge 1$ and $\eta > 0$, and suppose that*

$$A > \frac{\mu}{n(\kappa - \lambda + \gamma\lambda) \cdot \Big((\kappa-\lambda)R^n + \eta\Big)^{\gamma-1}}. \tag{3.1}$$

Then

$$w_{mid}(s) := \left\{ \frac{\mu}{n} - (1-\kappa)\gamma A \eta^{\gamma-1} \right\} \cdot R^n + A\eta^\gamma - A(\kappa R^n + \eta - s)^\gamma, \qquad s \in [\lambda R^n, \kappa R^n], \tag{3.2}$$

and

$$w_{out}(s) := \frac{\mu R^n}{n} - \gamma A \eta^{\gamma-1}(R^n - s), \qquad s \in [\kappa R^n, R^n], \tag{3.3}$$

satisfy

$$\partial_s w_{mid}(s) = \gamma A(\kappa R^n + \eta - s)^{\gamma-1} \qquad \textit{for all } s \in [\lambda R^n, \kappa R^n] \tag{3.4}$$

and

$$w_{mid}(\kappa R^n) = w_{out}(\kappa R^n) \tag{3.5}$$

as well as

$$\partial_s w_{mid}(\kappa R^n) = \partial_s w_{out}(\kappa R^n) \tag{3.6}$$

and

$$w_{mid}(\lambda R^n) < \lambda R^n \cdot \partial_s w_{mid}(\lambda R^n). \tag{3.7}$$

PROOF. The properties in (3.4), (3.5) and (3.6) can be verified by direct computation. To see that (3.1) ensures (3.7), we use (3.4) along with the inequalities $\eta \geq 0$ and $\gamma \geq 1$ in estimating

$$\begin{aligned}
& w_{mid}(\lambda R^n) - \lambda R^n \cdot \partial_s w_{mid}(\lambda R^n) \\
&\quad = \Big\{\frac{\mu}{n} - (1-\kappa)\gamma A \eta^{\gamma-1}\Big\} \cdot R^n + A\eta^\gamma - A\eta\Big((\kappa-\lambda)R^n + \eta\Big)^{\gamma-1} \\
&\qquad -(\kappa - \lambda + \gamma\lambda)R^n \cdot A\Big((\kappa-\lambda)R^n + \eta\Big)^{\gamma-1} \\
&\quad \leq \frac{\mu R^n}{n} - (\kappa - \lambda + \gamma\lambda)R^n \cdot A\Big((\kappa-\lambda)R^n + \eta\Big)^{\gamma-1} \\
&\quad < 0
\end{aligned}$$

due to (3.1). □

Here the inequality (3.7) provides a key for a linear extension of these candidates to globally C^1-regular functions, possible whenever the parameter η is suitably small.

Lemma 3.2 *Let $n \geq 1, R > 0, m > 1, \mu > 0, 0 < \lambda < \kappa < 1$ and $\gamma \geq 1$, and assume that $\eta > 0$ and $A > 0$ satisfy (3.1) and*

$$\eta^{\gamma-1} < \frac{\mu}{n\gamma A}. \tag{3.8}$$

Then with w_{mid} and w_{out} taken from Lemma 3.1,

$$k := \min\Big\{\tilde{k} > 0 \;\Big|\; \tilde{k}s \geq w_{mid}(s) \textit{ for all } s \in [\lambda R^n, \kappa R^n]\Big\} \tag{3.9}$$

and

$$s_0 := \min\Big\{s \in [\lambda R^n, \kappa R^n] \;\Big|\; ks = w_{mid}(s)\Big\} \tag{3.10}$$

are well-defined and have the properties that

$$\frac{\mu}{n} < k \leq \frac{\mu}{n\lambda} \tag{3.11}$$

as well as

$$s_0 \in (\lambda R^n, \kappa R^n). \tag{3.12}$$

Moreover,

$$\underline{w}(s) := \begin{cases} w_{in}(s) := ks, & s \in [0, s_0], \\ w_{mid}(s), & s \in (s_0, \kappa R^n), \\ w_{out}(s), & s \in [\kappa R^n, R^n], \end{cases} \tag{3.13}$$

defines a nonnegative function belonging to $W^{2,\infty}((0,R^n))$ *with* $\underline{w}_s(s) > 0$ *for all* $s \in [0, R^n]$.

PROOF. That both k and s_0 are well-defined is an immediate consequence of the continuity of w_{mid}, and the second inequality in (3.11) is directly implied by (3.9), (3.10) and the fact that $w_{mid} \le \frac{\mu R^n}{n}$. Next, introducing $\varphi(s) := ks - w_{mid}(s)$, $s \in [\lambda R^n, \kappa R^n]$, by definition of k and s_0 we see that

$$\varphi(s) \ge 0 \quad \text{for all } s \in [\lambda R^n, \kappa R^n] \qquad \text{and} \qquad \varphi(s_0) = 0, \tag{3.14}$$

which implies that we must have $s_0 > \lambda R^n$, because if we had $s_0 = \lambda R^n$, then since from Lemma 3.1 we know that the assumption (3.1) ensures that (3.7) holds, by definition of k we could infer that

$$\varphi'(\lambda R^n) = k - \partial_s w_{mid}(\lambda R^n) < k - \frac{w_{mid}(\lambda R^n)}{\lambda R^n} = k - \frac{k \cdot \lambda R^n}{\lambda R^n} = 0,$$

and that hence $\varphi < 0$ on $(s_0, s_0 + \delta)$ with some appropriately small $\delta > 0$.
Next, the nonnegativity feature in (3.14) together with (3.5) in particular ensures that

$$0 \le \varphi(\kappa R^n) = k \cdot \kappa R^n - w_{mid}(\kappa R^n) = k \cdot \kappa R^n - w_{out}(\kappa R^n)$$

and thus

$$k \ge \frac{w_{out}(\kappa R^n)}{\kappa R^n} = \frac{\mu}{n\kappa} - \frac{(1-\kappa)\gamma A \eta^{\gamma-1}}{\kappa} > \frac{\mu}{n\kappa} - \frac{\mu(1-\kappa)}{n\kappa} = \frac{\mu}{n}$$

thanks to (3.8). Furthermore, (3.8) shows that $s_0 < \kappa R^n$, for if we had $s_0 = \kappa R^n$, then due to (3.14) we should have $\varphi(\kappa R^n) = 0$ and $\varphi_s(\kappa R^n) \le 0$, in view of (3.5) and (3.6) meaning that

$$k = \frac{w_{out}(\kappa R^n)}{\kappa R^n} = \frac{\mu}{n\kappa} - \frac{\gamma A \eta^{\gamma-1}(1-\kappa)}{\kappa}$$

and that

$$0 \ge k - \partial_s w_{out}(\kappa R^n) = k - \gamma A \eta^{\gamma-1},$$

and that hence (3.8) would lead to the absurd conclusion that

$$0 \ge \frac{\mu}{n\kappa} - \frac{\gamma A \eta^{\gamma-1}(1-\kappa)}{\kappa} - \gamma A \eta^{\gamma-1} = \frac{\mu}{n\kappa} - \frac{\gamma A \eta^{\gamma-1}}{\kappa} > 0.$$

Thus knowing (3.12), we may go back to (3.14) to infer that necessarily $\varphi_s(s_0) = 0$, that is, $\partial_s w_{in}(s_0) = \partial_s w_{mid}(s_0)$, so that once more relying on (3.5) and (3.6) we readily obtain all claimed properties. □

Next turning our attention to the derivation of the subsolution features required in (2.12), in the following lemma we first concentrate on the intermediate region in (3.13), within which the intended inequality will turn out to hold under the assumption that besides the exponent γ, especially also the factor A in (3.2) is suitably large.

Lemma 3.3 *Let $n \geq 1, R > 0, m > 1, \mu > 0$ and $0 < \lambda < \kappa < 1$, and suppose that $\gamma > 1$, $\eta > 0$ and $A > 0$ are such that*

$$\gamma \geq \frac{m}{m-1} \tag{3.15}$$

and

$$A \leq \frac{\mu(1-\kappa)R^n}{4n\Big((\kappa-\lambda)R^n+\eta\Big)^\gamma} \tag{3.16}$$

and

$$A^{m-1} \leq \frac{\mu(1-\kappa)R^{n[-(m-1)\gamma+m-1+\frac{2}{n}]}}{2^m n^2 \gamma^{m-1}(\gamma-1)\kappa^{(m-1)\gamma-m+2-\frac{2}{n}}}, \tag{3.17}$$

as well as

$$\eta \leq \lambda R^n \tag{3.18}$$

and

$$\eta^{\gamma-1} \leq \frac{\mu}{4n\gamma A}. \tag{3.19}$$

Then for any $\varepsilon > 0$ fulfilling

$$\varepsilon \leq \gamma A \eta^{\gamma-1}, \tag{3.20}$$

the function w_{mid} from (3.2) has the property that

$$n^2 s^{2-\frac{2}{n}}(\partial_s w_{mid}+\varepsilon)^{m-1}\partial_{ss}w_{mid} + n w_{mid}\partial_s w_{mid} - \mu s \partial_s w_{mid} > 0 \qquad \textit{for all } s \in (\lambda R^n, \kappa R^n). \tag{3.21}$$

Proof. Since $nA\eta^\gamma > 0$, we see that within the range of s under consideration we have

$$\begin{aligned} n w_{mid}(s) - \mu s &= n \cdot \Big\{\frac{\mu}{n} - (1-\kappa)\gamma A \eta^{\gamma-1}\Big\} \cdot R^n + nA\eta^\gamma - nA(\kappa R^n + \eta - s)^\gamma - \mu s \\ &= \mu(R^n - s) - n(1-\kappa)\gamma A\eta^{\gamma-1}R^n + nA\eta^\gamma - nA(\kappa R^n+\eta-s)^\gamma \\ &\geq \mu(R^n - \kappa R^n) - n(1-\kappa)\gamma A\eta^{\gamma-1}R^n - nA(\kappa R^n + \eta - \lambda R^n)^\gamma \\ &= \mu(1-\kappa)R^n - n(1-\kappa)\gamma A\eta^{\gamma-1}R^n - nA\Big((\kappa-\lambda)R^n+\eta\Big)^\gamma \text{ for all } s \in (\lambda R^n, \kappa R^n). \end{aligned}$$

Here according to (3.19) we can estimate

$$\frac{n(1-\kappa)\gamma A\eta^{\gamma-1}R^n}{\mu(1-\kappa)R^n} = \frac{n\gamma A\eta^{\gamma-1}}{\mu} \leq \frac{1}{4},$$

whereas (3.16) warrants that also

$$\frac{nA\Big((\kappa-\lambda)R^n+\eta\Big)^\gamma}{\mu(1-\kappa)R^n} \leq \frac{1}{4},$$

so that altogether,

$$n w_{mid}(s) - \mu s \geq \frac{\mu(1-\kappa)R^n}{2} \qquad \text{for all } s \in (\lambda R^n, \kappa R^n). \tag{3.22}$$

Next, recalling (3.4) we compute

$$\partial_{ss} w_{mid}(s) = -\gamma(\gamma-1)A(\kappa R^n + \eta - s)^{\gamma-2}, \qquad s \in (\lambda R^n, \kappa R^n), \tag{3.23}$$

and moreover make use of (3.20) in estimating $\partial_s w_{mid}(s) \ge \gamma A \eta^{\gamma-1} \ge \varepsilon$ and hence

$$\partial_s w_{mid}(s) + \varepsilon \le 2\partial_s w_{mid}(s) \qquad \text{for all } s \in (\lambda R^n, \kappa R^n).$$

In conjunction with (3.23) and, particularly, the negativity of $\partial_{ss} w_{mid}$ on $(\lambda R^n, \kappa R^n)$ thereby implied, this ensures that

$$\begin{aligned} & n^2 s^{2-\frac{2}{n}} \Big(\partial_s w_{mid}(s) + \varepsilon\Big)^{m-1} \cdot \frac{\partial_{ss} w_{mid}(s)}{\partial_s w_{mid}(s)} \\ & \quad > \;\; 2^{m-1} n^2 (\kappa R^n)^{2-\frac{2}{n}} \cdot (\partial_s w_{mid}(s))^{m-2}\, \partial_{ss} w_{mid}(s) \\ & \quad = \;\; -2^{m-1} n^2 \gamma^{m-1} (\gamma-1) \kappa^{2-\frac{2}{n}} R^{2n-2} A^{m-1} (\kappa R^n + \eta - s)^{(m-1)\gamma - m} \qquad \text{for all } s \in (\lambda R^n, \kappa R^n). \end{aligned}$$

Here we may rely on the nonnegativity of $(m-1)\gamma - m$, as asserted by (3.15), to use (3.18) in verifying that

$$(\kappa R^n + \eta - s)^{(m-1)\gamma-m} \le (\kappa R^n + \eta - \lambda R^n)^{(m-1)\gamma-m} \le (\kappa R^n)^{(m-1)\gamma-m} \qquad \text{for all } s \in (\lambda R^n, \kappa R^n),$$

so that an application of (3.17) shows that

$$\begin{aligned} & n^2 s^{2-\frac{2}{n}} \Big(\partial_s w_{mid}(s) + \varepsilon\Big)^{m-1} \cdot \frac{\partial_{ss} w_{mid}(s)}{\partial_s w_{mid}(s)} \\ & \qquad > \;\; -2^{m-1} n^2 \gamma^{m-1} (\gamma - 1) \kappa^{(m-1)\gamma - m + 2 - \frac{2}{n}} R^{n\left[(m-1)\gamma - m + 2 - \frac{2}{n}\right]} A^{m-1} \\ & \qquad \ge \;\; -\frac{\mu(1-\kappa)R^n}{2} \qquad \text{for all } s \in (\lambda R^n, \kappa R^n). \end{aligned}$$

When combined with (3.22), once more by positivity of $\partial_s w_{mid}$ this entails (3.21). □

The corresponding subsolution features, both in the inner and in the outer region appearing in (3.13), quite easily result from the linear structure of the functions $\underline{w}_\eta$ in these parts. Therefore, completing our construction from this section essentially reduces to making sure that the requirements on the free parameters made in Lemma 3.1 and Lemma 3.2 in fact can simultaneously be fulfilled:

Lemma 3.4 *Let $n \ge 1, m > 1, \mu > 0$ and $0 < \lambda < \kappa < 1$. Then there exist $\gamma = \gamma(m, \kappa) > 1$ and $R_0 = R_0(n, m, \mu, \kappa, \lambda) > 0$ such that for any choice of $R \ge R_0$ one can find $\eta_0 = \eta_0(R, n, m, \mu, \kappa, \lambda) > 0$ and $A_0 = A_0(R, n, m, \mu, \kappa, \lambda) > 0$ with the property that for all $\eta \in (0, \eta_0)$ there exist $A_\eta > 0$ and $\varepsilon_\eta > 0$ such that $A_\eta < A_0$ and that the function $\underline{w} = \underline{w}_\eta$ defined through (3.13), with $w_{mid}, w_{out}, k = k_\eta$ and $s_0 = s_{0,\eta}$ as given in (3.2), (3.3), (3.9) and (3.10) with $A = A_\eta$, belongs to $W^{2,\infty}((0, R^n))$ with $\underline{w}_s > 0$ in $[0, R^n]$, and that whenever $\varepsilon \in (0, \varepsilon_\eta)$,*

$$n^2 s^{2-\frac{2}{n}} (\underline{w}_s + \varepsilon)^{m-1} \underline{w}_{ss} + n \underline{w}\, \underline{w}_s - \mu s \underline{w}_s > 0 \qquad \textit{for all } s \in (0, R^n) \setminus \{s_0, \kappa R^n\}. \tag{3.24}$$

Proof. Given $n \geq 1, m > 1, \mu > 0, \kappa \in (0,1)$ and $\lambda \in (0,\kappa)$, we fix $\gamma = \gamma(m,\kappa) > 1$ such that

$$\gamma \geq \frac{m}{m-1} \tag{3.25}$$

as well as

$$\gamma > \frac{(3+\kappa)\kappa}{(1-\kappa)\lambda} + 1 \tag{3.26}$$

and thereupon define

$$R_0 = R_0(n,m,\mu,\kappa,\lambda) := \sqrt{2^{2-m} n^{3-m} \gamma^{m-1} (\gamma-1) \kappa^{(m-1)\gamma-m+2-\frac{2}{n}} (\kappa-\lambda)^{-(m-1)\gamma} (1-\kappa)^{m-2} \mu^{m-2}}. \tag{3.27}$$

Then for $R \geq R_0$, we pick $\eta_0 = \eta_0(R,n,m,\mu,\kappa,\lambda) > 0$ small enough such that both

$$\eta_0 \leq \lambda R^n \tag{3.28}$$

and

$$\eta_0^{\gamma-1} \leq \frac{(\kappa-\lambda)^\gamma R^{n(\gamma-1)}}{(1-\kappa)\gamma} \tag{3.29}$$

hold, and for given $\eta \in (0,\eta_0)$ we let

$$A_\eta := \frac{(1-\kappa)\mu R^n}{4n \cdot \Big((\kappa-\lambda)R^n + \eta\Big)^\gamma} \tag{3.30}$$

and

$$\varepsilon_\eta := \gamma A_\eta \eta^{\gamma-1}, \tag{3.31}$$

noting that then, clearly,

$$A_\eta < A_0 := \frac{(1-\kappa)\mu}{4n(\kappa-\lambda)^\gamma R^{n(\gamma-1)}} \qquad \text{for all } \eta \in (0,\eta_0). \tag{3.32}$$

In order to verify that these choices moreover ensure simultaneous applicability of Lemma 3.2 and Lemma 3.3 for all $\eta \in (0,\eta_0)$ and each $\varepsilon \in (0,\varepsilon_\eta)$, we first observe that thanks to (3.26) we have $4\kappa < (1-\kappa)(\kappa-\lambda+\gamma\lambda)$ and therefore, by (3.30),

$$\begin{aligned}
\frac{A_\eta}{\mu} \cdot n(\kappa-\lambda+\gamma\lambda)\Big((\kappa-\lambda)R^n+\eta\Big)^{\gamma-1} &= \frac{(1-\kappa)(\kappa-\lambda+\gamma\lambda)R^n}{4\Big((\kappa-\lambda)R^n+\eta\Big)} \\
&> \frac{4\kappa R^n}{4\left((\kappa-\lambda)R^n+\eta\right)} \quad > 1,
\end{aligned}$$

meaning that indeed (3.1) is valid.
Next, (3.15), (3.16) and (3.18) are trivially asserted by (3.25), (3.30) and (3.28), whereas (3.19), itself obviously implying (3.8), results from (3.29), which due to (3.32) namely guarantees that

$$A_\eta \eta^{\gamma-1} < A_0 \cdot \frac{(\kappa-\lambda)^\gamma R^{n(\gamma-1)}}{(1-\kappa)\gamma} = \frac{\mu}{4n\gamma}. \tag{3.33}$$

Finally, once more by (3.32) we may use (3.27) along with our restriction $R \ge R_0$ to derive (3.17) by estimating

$$\begin{aligned} &\frac{2^m n^2 \gamma^{m-1}(\gamma-1)\kappa^{(m-1)\gamma-m+2-\frac{2}{n}}}{\mu(1-\kappa)R^{n[-(m-1)\gamma+m-1+\frac{2}{n}]}} \cdot A_\eta^{m-1} \\ &\qquad < \quad \frac{2^m n^2 \gamma^{m-1}(\gamma-1)\kappa^{(m-1)\gamma-m+2-\frac{2}{n}}}{\mu(1-\kappa)R^{n[-(m-1)\gamma+m-1+\frac{2}{n}]}} \cdot \Big\{ \frac{(1-\kappa)\mu}{4n(\kappa-\lambda)^\gamma R^{n(\gamma-1)}} \Big\}^{m-1} \\ &\qquad = \quad \frac{2^{2-m} n^{3-m} \gamma^{m-1}(\gamma-1)\kappa^{(m-1)\gamma-m+2-\frac{2}{n}}(1-\kappa)^{m-2}\mu^{m-2}}{(\kappa-\lambda)^{(m-1)\gamma}R^2} \qquad \le 1. \end{aligned}$$

As (3.31) implies (3.20), we may thus invoke both Lemma 3.2 and Lemma 3.3 to conclude that $\underline{w} = \underline{w}_\eta$ as in (3.13), with $k = k_\eta$ and $s_0 = s_{0,\eta}$ taken from (3.9) and (3.10) and $A = A_\eta$, has the claimed regularity and monotonicity properties and moreover satisfies the inequality in (3.24) for each $s \in (s_0, \kappa R^n)$, because $s_0 > \lambda R^n$ according to (3.12).
Apart from that, Lemma 3.2 warrants validity of (3.24) also for $s \in (0, s_0)$, because from (3.11) we know that

$$w_{in}(s) = ks > \frac{\mu}{n} s \qquad \text{for all } s \in (0, s_0)$$

and that thus

$$n\underline{w}(s)\underline{w}_s(s) - \mu s \underline{w}_s(s) = n\underline{w}_s(s) \cdot \Big(\underline{w}(s) - \frac{\mu}{n}s\Big) > 0 \tag{3.34}$$

for any such s, by linearity of $\underline{w}$ clearly implying (3.24) within this region.
Similarly, recalling (3.3) and (3.33) we find that also in the corresponding outer part,

$$\begin{aligned} w_{out}(s) - \frac{\mu}{n}s &= \Big\{ \frac{\mu R^n}{n} - \gamma A_\eta \eta^{\gamma-1}(R^n - s) \Big\} - \frac{\mu}{n}s \\ &= \Big(\frac{\mu}{n} - \gamma A_\eta \eta^{\gamma-1} \Big) \cdot (R^n - s) \quad > 0 \qquad \text{for all } s \in (\kappa R^n, R^n) \end{aligned}$$

and that consequently (3.34) and thus (3.24) hold throughout this interval as well. □

4 Persistently localized solutions. Proof of the main results

We are now in the position to apply Lemma 2.3, with subsolutions suitably selected from the ones in Lemma 3.4, to derive our main result on persistent localization by means of an extension argument appropriately arranged in such a manner that also in cases of finite-time singularity formation, the respective entire existence interval can be exhausted.

PROOF of Theorem 1.1. We let $\kappa := \sigma^n, \lambda := \theta^n$ and $l := \frac{\mu}{n\lambda}$, and observe that then $\kappa \in (0,1)$ and $\lambda \in (0, \kappa)$, and that our assumption (1.2) means that w_0 as in (2.2) satisfies

$$w_0(s) \ge ls \qquad \text{for all } s \in (0, \lambda R^n). \tag{4.1}$$

We then let $R_\star = R_\star(n, m, \mu, \sigma, \theta) := R_0(n, m, \mu, \kappa, \lambda)$ with $R_0(\cdot)$ taken from Lemma 3.4, and given $R \geq R_\star$ we also rely on Lemma 3.4 in defining $\eta_\star := \eta_0(n, m, \mu, \kappa, \lambda)$ and $A_\star := A_0(n, m, \mu, \kappa, \lambda)$. For $\eta \in (0, \eta_\star)$, we thereupon let $\underline{w}_\eta$ be as addressed in Lemma 3.4, and we claim that actually

$$w_0(s) \geq \underline{w}_\eta(s) \qquad \text{for all } s \in (0, R^n) \text{ and each } \eta \in (0, \eta_\star). \tag{4.2}$$

Indeed, this is evident from (4.1), because in view of (3.13), (3.12) and the second inequality in (3.11) we have

$$\underline{w}_\eta(s) \leq \frac{\mu}{n\lambda} \cdot s = ls \qquad \text{for all } s \in (0, \lambda R^n),$$

and because for larger s we can use that $\frac{1}{|\Omega|} \int_\Omega u_0 = \mu$ to trivially estimate

$$w_0(s) = \frac{\mu R^n}{n} \geq \underline{w}_\eta(s) \qquad \text{for all } s \in [\lambda R^n, R^n).$$

Now as a consequence of (4.2) when combined with an application of Lemma 2.3 to $\mathcal{F} := \{\underline{w}_\eta \mid \eta \in (0, \eta_\star)\}$, and to $\varepsilon_\star(\underline{w}_\eta) := \varepsilon_\eta$ for $\eta \in (0, \eta_\star)$, we see that the set

$$S := \Big\{ (T, u, v) \;\Big|\; T \in (0, \infty] \text{ and } (u, v) \text{ is a radial weak solution of (1.1) in } \Omega \times (0, T) \text{ such that } w \text{ from (2.1) satisfies } w(s, t) \geq \underline{w}_\eta(s) \text{ for all } s \in (0, R^n), t \in (0, T) \text{ and } \eta \in (0, \eta_\star) \Big\}$$

is not empty, and following [35] we introduce a partial ordering $\preceq$ on S by saying that $(T, u, v) \preceq (\tilde{T}, \tilde{u}, \tilde{v})$ if and only if $T \leq \tilde{T}$ and $(\tilde{u}, \tilde{v})|_{\Omega \times (0,T)} = (u, v)$. If I is any index set and $S_I := \{(T_\iota, u_\iota, v_\iota) \mid \iota \in I\}$ is totally ordered, then it is obvious that letting $T := \sup_{\iota \in I} T_\iota$ and $(u, v) := (u_\iota, v_\iota)$ in $\Omega \times (0, T_\iota)$ uniquely determines an upper bound (T, u, v) of S_I. Therefore, Zorn's lemma provides a maximal element (T_{max}, u, v) of S for which again by means of Lemma 2.3 we can derive (1.3):

In fact, assuming on the contrary that T_{max} be finite but that u be bounded in $\Omega \times (0, T_{max})$, from elliptic regularity theory applied to (2.9) we could infer boundedness of ∇v in $\Omega \times (0, T_{max})$, whereupon a standard testing procedure for (2.8) involving suitable regularized approximations of u^m as test functions would assert that $u^{m-1}\nabla u \in L^2(\Omega \times (0, T_{max}))$. Directly through (2.8), this would imply that $u_t \in L^2\left((0, T_{max}); (W^{1,2}(\Omega))^\star\right)$, which together with the boundedness of u would assert that u actually was uniformly continuous in $(0, T_{max})$ as an $L^2(\Omega)$-valued function. Therefore, with some nonnegative radial $\tilde{u}_0 \in L^\infty(\Omega)$ we would have $u(\cdot, t) \to \tilde{u}_0$ in $L^2(\Omega)$ as $t \nearrow T_{max}$, where from the definition of S we clearly obtain that also $\int_0^{s^{\frac{1}{n}}} \rho^{n-1} \tilde{u}_0(\rho) d\rho \geq \underline{w}_\eta(s)$ for all $s \in (0, R^n)$ and $\eta \in (0, \eta_\star)$. Now Lemma 2.3 would once again apply so as to yield $T > 0$ and a radial weak solution $(\tilde{u}, \tilde{v})$ of (1.1) in $\Omega \times (0, T)$ with $\tilde{u}|_{t=0} = \tilde{u}_0$, additionally fulfilling $\int_0^{s^{\frac{1}{n}}} \rho^{n-1} \tilde{u}(\rho, t) d\rho \geq \underline{w}_\eta(s)$ for all $s \in (0, R^n)$, $t \in (0, T)$ and $\eta \in (0, \eta_\star)$. In consequence,

$$(U, V)(\cdot, t) := \begin{cases} (u, v)(\cdot, t), & t \in (0, T_{max}), \\ (\tilde{u}, \tilde{v})(\cdot, t - T_{max}), & t \in [T_{max}, T_{max} + T), \end{cases}$$

would define an extension of (u, v) to a radial weak solution (U, V) of (1.1) in $\Omega \times (0, T_{max} + T)$ such that $\int_0^{s^{\frac{1}{n}}} \rho^{n-1} U(\rho, t) d\rho \geq \underline{w}_\eta(s)$ for all $s \in (0, R^n)$, any $t \in (0, T_{max} + T)$ and each $\eta \in (0, \eta_\star)$, contradicting the maximality of (T_{max}, u, v).

In order to finally derive (1.4), once more relying on the definition of S we now exploit the inequality

$$w(s,t) \ge \underline{w}_\eta(s), \qquad s \in (0,R^n),\ t \in (0,T_{max}), \tag{4.3}$$

exclusively for values $s \in (\kappa R^n, R^n)$, as for which, namely, from (3.13), Lemma 3.1 and Lemma 3.4 we know that

$$\underline{w}_\eta(s) = \frac{\mu R^n}{n} - \gamma A_\eta \eta^{\gamma-1}(R^n - s) \ge \frac{\mu R^n}{n} - \gamma A_\star \eta^{\gamma-1} R^n \to \frac{\mu R^n}{n} \qquad \text{as } \eta \searrow 0,$$

because $\gamma > 1$ by Lemma 3.4. Thus, (4.3) entails that for all $t \in (0, T_{max})$,

$$w(s,t) \ge \frac{\mu R^n}{n} \qquad \text{for all } s \in (\kappa R^n, R^n),$$

which we may combine with the opposite inequality $w \le \frac{\mu R^n}{n}$, as implied throughout $(0,R^n)\times(0,T_{max})$ due to the mass conservation property (2.10). We thereby conclude that for each $t \in (0,T_{max})$ we have

$$w(s,t) = \frac{\mu R^n}{n} \qquad \text{for all } s \in (\kappa R^n, R^n)$$

and hence

$$u(r,t) = n w_s(r^n, t) = 0 \qquad \text{for a.e. } r \in (\kappa^{\frac{1}{n}} R, R) = (\sigma R, R),$$

as claimed. □

Our application of this to essentially bell-shaped initial data with compact support, finally, is straightforward:

Proof of Corollary 1.2. When translated to the variables in (2.2), the assumed monotonicity of u_0 ensures that w_0 is concave on $(0,R^n)$, while (1.5) asserts that $\operatorname{supp} \partial_s w_0 \subset [0, \theta^n R^n]$ and hence $w_0 \equiv \frac{\mu R^n}{n}$ in $[\theta^n R^n, R^n]$. In combination, these properties can easily be seen to guarantee that $w_0(s) \ge \frac{\mu}{n\theta^n} \cdot s$ for all $s \in (0, \theta^n R^n)$, which is equivalent to (1.2). The claim therefore results from Theorem 1.1. □

5 Discussion

An interesting question complementary to the one addressed in our Theorem 1.1 is whether for suitable large classes of solutions also lower estimates for the corresponding positivity sets can be derived. In view of precedents concerned with large time behavior of bounded solutions to related problems in the literature ([20], [28]), it is likely to be expected that ω-limit sets of trajectories, which are global and bounded, should contain a reasonable regular nontrivial steady state, and that hence the corresponding support can at least not shrink to single points asymptotically.

Along with the seemingly yet more delicate question, whether some exploding solutions might have their support collapse into a singleton, however, detailing this in the current problem setting based on the approximation in (2.4) would go beyond the scope of the present manuscript.

Now let us shortly repeat the rescaling arguments as given in [27] for (degenerate) chemotaxis-systems of the form

$$\partial_t u = D\nabla \left(u^{m-1}\nabla u\right) - \chi\nabla(u\nabla v) \quad , \quad \partial_t v = \gamma\Delta v - \eta v + \beta u \quad \text{in } \Omega = B_R(0) \subset \mathbb{R}^n \; , \tag{5.1}$$

with Neumann-type boundary conditons and positive parameters $D, \chi, \gamma, \eta, \beta$. After rescaling space, we obtain that $\partial_t u = \nabla \left(u^{m-1}\nabla u\right) - \frac{\chi}{D}\nabla(u\nabla v)$ and $\partial_t v = \frac{\gamma}{D}\Delta v - \eta v + \beta u$.
Setting $\tilde{v} := \frac{\chi}{D}v$, we get $\partial_t u = \nabla \left(u^{m-1}\nabla u\right) - \nabla(u\nabla\tilde{v})$ and $\partial_t\tilde{v} = \frac{\gamma}{D}\Delta\tilde{v} - \eta\tilde{v} + \frac{\chi\beta}{D}u$.
Now let $\bar{w} := \frac{1}{|\Omega|}\int_\Omega w \, dx$, then $\bar{u}(t) = \bar{u}_0$ and $\partial_t\bar{\tilde{v}} + \eta\bar{\tilde{v}} = \frac{\chi\beta}{D}\bar{u}_0$.
Expressing $\beta = \alpha\gamma$ and defining $\hat{v} := \tilde{v} - \bar{\tilde{v}}$, we obtain $\frac{D}{\gamma}\left(\partial_t + \eta\right)\hat{v} = \Delta\hat{v} + \chi\alpha(u - \bar{u}_0)$.
Assuming $\gamma >> D$ we can approximate the last equation by $0 = \Delta\hat{v} + \chi\alpha\left(u - \bar{u}_0\right)$.
Since $\nabla\hat{v} = \nabla\tilde{v}$, the rescaled system reads, after renaming $\hat{v}$ back to v:

$$\partial_t u = \nabla \left(u^{m-1}\nabla u\right) - \nabla(u\nabla v) \quad , \quad 0 = \Delta v + \chi\alpha\left(u - \bar{u}_0\right) \; . \tag{5.2}$$

This is the version we have been looking at for $\chi \cdot \alpha = 1$ and $\bar{u}_0 = \mu$.
Now let $u^* = \frac{u}{\bar{u}_0}$ and $v^* = \frac{v}{\chi\alpha\bar{u}_0}$, then

$$\partial_t u^* = \bar{u}_0^{m-1}\nabla\left(\left(u^*\right)^{m-1}\nabla u^*\right) - \chi\alpha\bar{u}_0\nabla\left(u^*\nabla v^*\right) \quad , \quad 0 = \Delta v^* + u^* - 1 \; . \tag{5.3}$$

This is the version considered in [27] for general $\chi \cdot \alpha$.

Recently there has been a renewed interest in the literature to understand steady states of (5.1) and further generalized versions of this system, a by now classic contribution being [44]. One question of interest concerns compactness and connectedness of the support of steady state solutions, respectively its structure and size. In this context, also so-called aggregation equations are analyzed, which relate to our system in the following way. Considering (5.1) in $\mathbb{R}^n$, but setting $0 = \Delta v - v + u$, i.e. $v = (Id - \Delta)^{-1}u$, one obtains

$$\partial_t u = D\nabla\left(u^{m-1}\nabla u\right) - \chi\nabla\left(u\nabla\left(G * u\right)\right) \; , \tag{5.4}$$

where G is the Besselkernel of order 2. Rescaling time we get

$$\partial_t u = \varepsilon\nabla\left(u^{m-1}\nabla u\right) - \nabla\left(u\nabla\left(W * u\right)\right) \; , \tag{5.5}$$

where $W = G/||G||_{L^1}$ has L^1-norm one, and $\varepsilon = \frac{D}{\chi||G||_{L^1}}$. The corresponding energy in R^n reads

$$\mathcal{E}[u] = \int \frac{\varepsilon(m-1)}{m}u^m(x)\,dx - \frac{1}{2}\int\int W(x-y)u(y)u(x)\,dy\,dx \; . \tag{5.6}$$

Similarly, with the respective adapted kernels, also bounded domains can be considered. The existence of minimizers of such energies and their structure are of interest, also in the context of steady states of equations of type (5.4) for a variety of kernels W. Therefore we give a glimpse on some of the related literature here. Before doing so, let us briefly mention that degenerate diffusion equations with non-local aggregation effects have been of interest for a long time in mathematical ecology, see e.g. [40] for one of many examples.

In [2], equation (5.4) and also more general equations and kernels, are considered in bounded domains as well as in $\mathbb{R}^n$. Local well-posedness is proved in bounded domains for $n \geq 2$. Subcritical problems are globally well-posed and a critical mass is obtained, which sharply divides the possibility of finite time blow-up and global existence. This is well known for $m = 1$, which was first rigorously proved in [27], where the sharp optimal parameter for this dichotomy can be directly read of the presented estimates.
In connection with the Bessel potential (but also for other G), in [2] existence and uniqueness of weak solutions was proved for $n > 3$. Existence and uniqueness w.r.t. entropy solutions were considered in [5], and for $n = 1$ uniqueness was proved in [7].
Results on related stationary solutions of (5.4) in $\mathbb{R}^n$, and thus on critical points of the associated free energy functional (5.6) and compact support in the context of our system are mostly considered for the Newtonian or regularized Newtonian potential - which means no decay of v - rather than the Bessel potential. For $m > 2 - \frac{2}{n}$ and $n \geq 3$ there exsits a unique radially symmetric stationary solution, which is monotonically decreasing and has compact support, see [32] and references therein, and [14]. For $n = 2$ and the Newtonian potential, existence of a unique compactly supported stationary solution was shown in [11]. For $n \geq 3$ in the supercritical case $0 < m < 2 - \frac{2}{n}$ see e.g. [8], and [9], [13] for further cases, as well as [29], also for some literature before 2017.

Connections to constrained aggregation equations, the respective constrained interaction energy and their relations to shape optimization problems have been recently considered in [6] and [18], where also uniform bounds on the support of minimizers were proved. The geometry of minimizers for general mildly repulsive interaction potentials at the origin was classified in [12].

Varying the classical simplified Keller-Segel-system, $m = 1$, by $m > 1$ is not the only way to obtain localization of (the main amount of) mass in patches, since for these patterns to occur, a balance between repulsion and attraction is needed. In [46] and [47], the regularized system $\partial_t u = \Delta u - \nabla(g_\varepsilon(u)\nabla v)$, $0 = \Delta v + u$, was considered in $\mathbb{R}^2$, with e.g. g_ε being a saturating function. For the steady state solution $\bar{u}$, $\bar{u} = \frac{\bar{U}}{\varepsilon}$ and $\bar{V} = \log(\bar{U})$, in the radial symmetric setting $\bar{V}$ fulfills the Emden equation. The solution u can be decomposed in a regular part and a set of concentration regions of order $\sqrt{\varepsilon}$ where an amount of mass of order one concentrates. Detailed dynamics/interactions of these patchy concentration regions, their respective mass and the regular part are derived. For $\varepsilon \to 0$ and regions with a high density of mass it is proved that the solution makes a transition between the blowing up behavior and the "quasi-steady behavior". The size of the transition region is described as well.

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