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by

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Abstract

We study a rough difference equation on a discrete time set, where the driving Hölder rough path is a realization of a stochastic process. Using a modification of Davie's approach [6] and the discrete sewing lemma, we derive norm estimates for the discrete solution. In particular, when the discrete time set is regular, the system generates a discrete random dynamical system. We also generalize a recent result in [10] on the existence and upper semi-continuity of a global random pullback attractor under the dissipativity and the linear growth condition for the drift.

Keywords: rough path theory, rough integrals, rough differential equations, rough difference equations, random dynamical systems, random attractors, stochastic perturbation, Euler scheme, numerical random attractors, random attractor approximation.

1 Introduction

This work aims to provide a systematic approach to study the numerical attractor for the following rough differential equation

$$dy_t = f(y_t)dt + g(y_t)dx_t, \quad y_0 \in \mathbb{R}^d \tag{1.1}$$

where f, g satisfy certain regularity conditions, and $x \in C^\nu(\mathbb{R}^m)$ for $\nu \in (\frac{1}{3}, 1)$ is a $\nu-$ Hölder continuous realization of a stochastic process that can be lifted into a rough path $\mathbf{x} = (x, \mathbb{X})$. It is well-known that such an equation can be interpreted and solved by Lyons' rough path theory [17] or its reformulations [13], [14]. Since the drift f is often unbounded, the strategy is to solve the pure rough differential equation $z = g(z)dx$ for g of bounded or linear form, and then use the Doss-Sussmann technique to transform system (1.1) to an ordinary differential equation and then solve it on each interval of consecutive stopping times (see e.g. [19], [8], [10] and the references therein). It is often assumed that g is linear or $g \in C_b^3$ (at least in $C^{2+\gamma}$ for a certain $\gamma \in (0,1)$), so that one can prove the existence and uniqueness of solution as well as its norm estimates on any time interval $[0,T]$. When g is neither linear nor bounded, the solution is proved in [6] and [16] to exists upto a stopping time and can blow after that, thus in general it might not be extended into an arbitrary interval.

The Euler scheme for rough differential equation is first studied under the frame work of rough path theory for discrete time sets in the classical work of Davie [6] for pure rough equation and later in [13], [15] (see also recent works in [3]). Roughly speaking, this approach serves as an alternative to investigate the existence and uniqueness theorem for solution of (1.1) in which the priori estimates evaluate the solution norm to ensure that the solution do not blow in finite time, and the discretization scheme as well as its solution norm estimates are used to support the proof.

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Another approach in [10] considers the Euler scheme of (1.1) on a discrete time set $\Pi = \{t_k\}_{k\in\mathbb{N}}$ defined by

$$y_{t_0} \in \mathbb{R}^d, \quad y_{t_{k+1}} = y_{t_k} + f(y_{t_k})(t_{k+1} - t_k) + F_{t_k,t_{k+1}}(y, \mathbf{x}), \quad \text{for} \quad k = 0, 1, \dots \tag{1.2}$$

where

$$F_{s,t}(y, \mathbf{x}) = F_{s,t} = \begin{cases} g(y_s)x_{s,t} & \text{for Young difference equation} \\ g(y_s)x_{s,t} + Dg(y_s)g(y_s)\mathbb{X}_{s,t} & \text{for rough difference equation.} \end{cases} \tag{1.3}$$

The solution estimate for the discrete system (1.2) is then studied via a comparison to the solution of the continuous system (1.1), using a cut-off technique, where it is proved in [10] that the solution norm of system (1.1) can be estimated for unbounded and one-sided Lipschitz continuous f, provided its linear growth in the perpendicular direction.

The asymptotic dynamics of system (1.1) is studied under the framework of random dynamical systems [1], since one can prove (see [2]) that (1.1) generates a continuous random dynamical system φ. Under an additional assumption on the dissipativity of f, it is shown in [7], [10] that φ admits a random pullback attractor $\mathcal{A}$. It is then natural to ask the question on how one can approximate $\mathcal{A}$ by a discretization scheme. When the discrete time set Π is regular, i.e. $\Pi = \Pi^\Delta = \{k\Delta\}_{k\in\mathbb{N}}$ and g is bounded, system (1.2) is proved in [10] to generates a discrete random dynamical system φ^Δ which admits a discrete random pullback attractor $\mathcal{A}^\Delta$. The upper semi-continuity of the numerical attractor $\mathcal{A}^\Delta$ to $\mathcal{A}$ as the step size Δ tends to zero is only affirmative under the bounded condition of both coefficient functions f and g.

In this paper, we would like to present a direct approach for studying the discrete system (1.2), without considering the limiting equation (1.1), the approach is then similar to the classical one [6] but for the mixed equation (1.1) with the dt part. To do that, we impose the following conditions for the coefficient funtions f and g as follows.

($\mathbf{H_f}$) $f : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function of linear growth, i.e. there exists a constant $C_f > 0$ such that

$$\|f(y)\| \leq C_f\|y\| + \|f(0)\|, \quad \forall y \in \mathbb{R}^d, \tag{1.4}$$

where $f(0)$ is the value of f evaluated at the vector $0 \in \mathbb{R}^d$;

($\mathbf{H_g}$) regarding to (1.3),

- ($\mathbf{H}_g^y$): in Young case, g is in $C^1(\mathbb{R}^d, \mathbb{R}^m)$ with its derivative bounded by a constant C_g and globally Lipschitz continuous;

- ($\mathbf{H}_g^r$): in rough case, g either belongs to $C^2(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d))$ such that it is bounded together with its derivatives

$$\|g\|_\infty, C_g := \max\Big\{\|Dg\|_\infty, \|D^2g\|_\infty\Big\} < \infty, \tag{1.5}$$

  or it has a linear form $g(y) = Cy + g(0)$, where $C \in \mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d))$ such that $\|C\| \leq C_g$.

Our main results (Theorem 3.3, Theorem 4.2, Theorem 5.3) can be summarized as follows.

Theorem 1.1 *Under assumptions (*$\mathbf{H_f}$*), (*$\mathbf{H_g}$*), there exist polynomials* $\xi_i(L_g \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert_{p,\Pi[a,b]}), i = 1, 2, 3$ *of* $L_g \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert_{p,\Pi[a,b]}$ *such that the solution of the rough difference equation* (1.2) *satisfies*

$$(i), \|y\|_{\infty,\Pi[a,b]} \leq \|y_a\|e^{\xi_1(L_g \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert_{p,\Pi[a,b]})} + C_0e^{\xi_2(L_g \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert_{p,\Pi[a,b]})} - C_0;$$
$$(ii), \|(y, R^y)\|_{p,\Pi[a,b]} \leq (\|y_a\| + C_0)e^{\xi_3(L_g \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert_{p,\Pi[a,b]})}.$$

In case g is bounded, ξ_1 can be chosen independently of $\mathbf{x}$*.*

Theorem 1.2 *Under the assumptions* ($\mathbf{H_f}$), ($\mathbf{H_g}$), *assume further the dissipativity condition for* f

$$\exists c,d>0: \quad \langle y, f(y)\rangle \leq c - d\|y\|^2, \quad \forall y\in\mathbb{R}^d.$$

Consider system (1.2) *with the regular time set* Π^Δ, *where* $\Delta\in(0,1)$ *satisfies the inequality*

$$0<\Delta<1\wedge\frac{d}{2C_f^2}\wedge\frac{1}{2d}.$$

Then under the bounded moment condition on Π^Δ, *i.e.*

$$\mathbb{E}\,\|\mathbf{x}(\cdot)\|^k_{p,\Pi[a,b]}<\infty, \quad \forall k\in\mathbb{N}, \quad \forall 0<a<b, a,b\in\Pi^\Delta,$$

and for L_g *small enough (independent of* Δ*), the generated discrete random dynamical system* φ^Δ *of* (1.2) *admits a random pullback attractor* $\mathcal{A}^\Delta$. *In relation to system* (1.1), *if*

$$\mathbb{E}\,\|\mathbf{x}(\cdot)\|^k_{p,[a,b]}<\infty, \quad \forall k\in\mathbb{N}, \quad \forall 0<a<b$$

then there exists also a random attractor $\mathcal{A}$ *for* φ *of* (1.1) *to which the numeric attractor* $\mathcal{A}^\Delta$ *converges to in the Hausdorf semi-distance, i.e.* $d_H(\mathcal{A}^\Delta|\mathcal{A})\to 0$ *as* $\Delta\to 0$, *a.s.*

Thanks to the fundamental sewing lemma [14] for the discrete time set which is introduced previously in [6] or recently in [3], we are able to derive, in the first main result, direct estimates for solution norms of the discrete system (1.2) by using similar techniques in [8] and by constructing a modified version of greedy sequences of stopping times [4] for the discrete framework. Note that for the discrete system (1.2), the above assumption on g is relaxed only a little ($g\in C_b^2$ or of linear form) and can deal with quite general g in the Young case. Meanwhile, the assumption on linear growth f (though stronger than the one in [10]) helps estimate the supremum norm $\|f(y)\|_\infty$ by $C_f\|y\|_p$ and make the construction of stopping times extend to infinity, and eventually yields a p-variation norm estimate of the discrete solution dependent on only global parameters C_f, L_g.

The discrete sewing lemma also plays the most important role in the second main result by showing the uniform boundedness of the numerical pullback absorbing set w.r.t. the time step Δ, which leads to the upper semi-continuity of the numerical attractor. This is the advancement of our techniques compared to the previous ones in [10], where the pullback absorbing set can be very large as Δ tends to zero (even with g bounded), thus one also needs f to be bounded in order to obtain the upper semi-continuity for $\mathcal{A}^\Delta$. Our results also hold for rough equation (1.1) with lower regularity of the driving path x, in that case F in (1.3) should be modified and complexity in technical details should also be expected.

2 Discrete framework

2.1 Discrete settings

Since we investigate discrete approximation of solutions of rough differential equations on $[0,T]$, we will have to deal with discrete functions on $[0,T]$: functions defined on a finite set of points of $[0,T]$. In this subsection we present some basic notions of discrete functions.

Let $[a,b]$ be a closed interval of $\mathbb{R}$, and $\Pi=\{t_i: 0\leq i\leq n, a=t_0<t_1<\ldots<t_n=b\}$ be an arbitrary (finite) set of points of $[a,b]$. Since Π actually makes a partition of $[a,b]$ into $\cup_{i=0}^{n-1}[t_i,t_{i+1}]=[a,b]$, with an abuse of language we sometimes call Π a (finite) partition of $[a,b]$. The number $|\Pi|:=\max_{0\leq i\leq n-1}(t_{i+1}-t_i)>0$ is called *mesh* of the finite partition Π. By *a discrete* *function* defined on Π, we mean a map $y: \Pi \to \mathcal{B}, \quad \Pi \ni t_i \mapsto y_{t_i} \in \mathcal{B}$ on a normed space B. For discrete functions we introduce various norms which are natural discrete versions of the continuous ones. Namely,

$$\begin{aligned}
\|y\|_{\infty,\Pi} &:= \sup_{t_i \in \Pi} \|y_{t_i}\|; \\
\|\!|y|\!\|_{p,\Pi} &:= \sup_{t_i^* \in \Pi,\, 0 \le i \le r,\, t_0^* < t_1^* \ldots < t_r^*,\, r \le n} \left(\sum_{i=0}^{r-1} \|y_{t_i^*} - y_{t_{i+1}^*}\|^p \right)^{1/p}; \\
\|y\|_{p,\Pi} &= \|y_a\| + \|\!|y|\!\|_{p,\Pi}.
\end{aligned}$$

Clearly $\|\cdot\|_{\infty,\Pi}$ and $\|\cdot\|_{p,\Pi}$ are norms on the space of discrete function determined on Π, whereas $\|\!|\cdot|\!\|_{p,\Pi}$ is a semi-norm.

For a discrete function F on $\Delta\Pi := \{(s,t) \in \Pi^2 : s \le t\}$, we also define discrete supremum and p-variation norm by

$$\begin{aligned}
\|\!|F|\!\|_{p,\Pi} &:= \sup_{t_i^* \in \Pi,\, 0 \le i \le r,\, t_0^* < t_1^* \ldots < t_r^*,\, r \le n} \left(\sum_{i=0}^{r-1} |F_{t_i^*, t_{i+1}^*}|^p \right)^{1/p}, \\
\|F\|_{\infty,\Pi} &:= \sup_{a \le s \le t \le b} \|F_{s,t}\|.
\end{aligned}$$

The notion of a control function also has its discrete counterpart.

Definition 2.1 *A non negative function ω defined on $\Delta\Pi := \{(s,t) \in \Pi^2 | s \le t\}$ is called a* discrete control function *on Π if it vanishes on the diagonal, i.e. $\omega_{s,s} = 0, \forall s \in \Pi$, and is superadditive, i.e. for all $s \le u \le t$ in Π*

$$\omega_{s,u} + \omega_{u,t} \le \omega_{s,t}.$$

As an example, if F vanishes on the diagonal of Π, $\omega_{s,t} := \|\!|F|\!\|^q_{q,\Pi[s,t]}, (s,t) \in \Delta\Pi$ is a discrete control. If ω is a discrete control on $[a,b]$ and $|y_{t_k} - y_{t_l}| \le \omega^{1/p}_{t_k,t_l}$ for all $t_k, t_l \in \Pi$, $p \ge 1$ then

$$\|\!|y|\!\|_{p,\Pi[a,b]} \le \omega^{1/p}_{a,b}.$$

Furthermore, if $y_\cdot$ is a continuous function of bounded p-variation on $[a,b]$, and $\Pi[a,b]$ is a finite partition of $[a,b]$ then the function y restricted on $\Pi[a,b]$ is a discrete function and we have the following relation between continuous and discrete norms of y:

$$\|y\|_{\infty,\Pi[a,b]} \le \|y\|_{\infty,[a,b]}; \quad \|y\|_{p,\Pi[a,b]} \le \|y\|_{p,[a,b]}.$$

The notion of discrete function and discrete control function can be easily generalized for the case of arbitrary (not necessarily finite) subset $\Pi \subset [a,b]$.

2.2 Discrete rough paths

For given $[a,b]$ and Π a (finite) discrete time set in $[a,b]$ as above, let $x_\cdot : \Pi \to \mathbb{R}^d$ and $\mathbb{X}(\cdot,\cdot) : \Pi^2 \to \mathbb{R}^{d\times d}$ be discrete functions defined on Π and Π^2 respectively. One then says that x can be lifted to a discrete rough path $\mathbf{x} = (x, \mathbb{X})$ if it satisfies Chen's relation

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = x_{s,u} \otimes x_{u,t}, \quad s \le u \le t;\ s,u,t \in \Pi. \tag{2.1}$$

One can then furnish $\mathbf{x}$ with a semi-norm

$$\|\mathbf{x}\|_{p,\Pi[a,b]} = \left(\|x\|^p_{p,\Pi[a,b]} + \|\mathbb{X}\|^{p/2}_{p/2,\Pi[a,b]} \right)^{1/p}. \tag{2.2}$$

Denote by $T^2(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes R^d)$ the truncated step-2 tensor algebra, then $T^2(\mathbb{R}^d)$ is a Banach space with the norm $|\omega| := \max_{k=0,1,2} \|\pi_k(\omega)\|$ where $\|\pi_k(\omega)\|$ is the Euclidean norm on $(\mathbb{R}^d)^{\otimes k}$. Define $T^2_1(\mathbb{R}^d) = \{\omega \in T^2(\mathbb{R}^d) : \pi_0(\omega) = 1\}$ together with a $\bullet$ operator

$$(1, g^1, g^2) \bullet (1, h^1, h^2) = (1, g^1 + h^1, g^2 + h^2 + g^1 \otimes h^1), \tag{2.3}$$

for any $\mathbf{g} = (1, g^1, g^2), \mathbf{h} = (1, h^1, h^2)$. Then $(T^2_1(\mathbb{R}^d), \bullet)$ is a group with the identity element $\mathbf{1} = (1, 0, 0)$, and each element $\mathbf{g}$ has its inverse element $\mathbf{g}^{-1} = (1, -g^1, -g^2 - g^1 \otimes g^1)$. In addition, for a rough path $\mathbf{x}$ one can easily check from Chen's relation (2.1) that

$$(1, x_{s,u}, \mathbb{X}_{s,u}) \bullet (1, x_{u,t}, \mathbb{X}_{u,t}) = (1, x_{s,t}, \mathbb{X}_{s,t}), \quad \forall s \le u \le t;\ s,u,t \in \Pi.$$

Also, one can view $\mathbf{x}_t = (1, x_{0,t}, \mathbb{X}_{0,t})$ as a map from Π to $T^2_1(\mathbb{R}^d)$.

2.3 Discrete sewing lemma

In this section we fix $[a, b]$ and a finite partition $\Pi = \{t_i : 0 \le i \le n, a = t_0 < t_1 < \ldots < t_n = b\}$ on $[a, b]$. The following lemma is the main result of this section. It is actually an algebraic result and provides us with an effective tool for investigation of discretized rough differential equations. For a version of Hölder norm, see [15].

Lemma 2.2 (Discrete sewing lemma) *Let $\Pi = \{t_i : 0 \le i \le n, a = t_0 < t_1 < \ldots < t_n = b\}$ be a finite partition of $[a, b]$ and F be an function defined on $\Delta\Pi$, vanished on the diagonal, i.e. $F_{s,s} = 0, \forall s \in \Pi$. Put*

$$\begin{aligned} (\delta F)_{s,u,t} &:= F_{s,t} - F_{s,u} - F_{u,t}, \quad s \le u \le t;\ s,u,t \in \Pi, \\ I_{k,l} &:= \sum_{k \le j \le l-1} F_{t_j,t_{j+1}} - F_{t_k,t_l}, \quad t_k \le t_l \in \Pi. \end{aligned}$$

Assume that for a discrete control ω on Π and a number $\lambda > 1$ we have for all $s \le u \le t$ in Π the following inequality

$$\|(\delta F)_{s,u,t}\| \le \omega^\lambda_{s,t}. \tag{2.4}$$

Then there exists a constant $K > 0$ depending only on λ such that

$$\|I_{k,l}\| \le K \omega^\lambda_{t_k,t_l}, \quad \forall\, t_k \le t_l \in \Pi. \tag{2.5}$$

Proof: See e.g. [3]. □

Corollary 2.3 *Let $\Pi = \{t_i : 0 \le i \le n, a = t_0 < t_1 < \ldots < t_n = b\}$ be a finite partition of $[a, b]$ and F be an function defined on $\Delta\Pi$, vanished on the diagonal, i.e. $F_{s,s} = 0, \forall s \in \Pi$. Put*

$$\begin{aligned} (\delta F)_{s,u,t} &:= F_{s,t} - F_{s,u} - F_{u,t}, \quad s \le u \le t;\ s,u,t \in \Pi, \\ I_{k,l} &:= \sum_{k \le j \le l-1} F_{t_j,t_{j+1}} - F_{t_k,t_l}, \quad t_k \le t_l \in \Pi. \end{aligned}$$

Assume that for a finite set $\mathcal{S}$ of discrete controls ω on Π and a number $\lambda > 1$ we have for all $s \le u \le t$ in Π the following inequality

$$\|(\delta F)_{s,u,t}\| \le \sum_{\omega \in \mathcal{S}} \omega^{\lambda}_{s,t}. \tag{2.6}$$

Then there exists a constant $K > 0$ depending only on λ such that

$$\|I_{k,l}\| \le K \sum_{\omega \in \mathcal{S}} \omega^{\lambda}_{t_k,t_l}, \quad \forall\, t_k \le t_l \in \Pi. \tag{2.7}$$

Proof: Since $\lambda > 1$, observe from (2.6) that

$$\|(\delta F)_{s,u,t}\| \le \sum_{\omega \in \mathcal{S}} \omega^{\lambda}_{s,t} \le \Big(\sum_{\omega \in \mathcal{S}} \omega_{s,t}\Big)^{\lambda}.$$

The proof then follows directly from Lemma 2.2 and Jensen inequality that

$$\|I_{k,l}\| \le K\Big(\sum_{\omega \in \mathcal{S}} \omega_{t_k,t_l}\Big)^{\lambda} \le K|\mathcal{S}|^{\lambda-1}\,\Big(\sum_{\omega \in \mathcal{S}} \omega^{\lambda}_{t_k,t_l}\Big).$$

□

Corollary 2.4 *Let $\Pi = \{t_i : 0 \le i \le n, a = t_0 < t_1 < \ldots < t_n = b\}$ be a finite partition of $[a,b]$. Consider a discrete system defined on Π*

$$y_{t_{j+1}} = y_{t_j} + F_{t_j,t_{j+1}} + \epsilon_{t_j,t_{j+1}}, \quad y_{t_0} = y^* \in \mathbb{R}^d, \quad j = 0,1,\ldots,n-1, \tag{2.8}$$

where $F : \Pi^2 \to \mathbb{R}^d$, $\epsilon : \Pi^2 \to \mathbb{R}^d$. Assume that
(i) There exists a discrete control ω such that (2.4) is satisfied for F;
(ii) There exists a discrete control function $\omega^{(0)}$ such that $|\epsilon_{t_j,t_{j+1}}| \le \omega^{(0)}_{t_j,t_{j+1}}$.
Then there exists a constant $K > 0$ such that for any $r \ge 1$

$$\|y\|_{r,\Pi[a,b]} \le K\omega^{\lambda}_{a,b} + \|F\|_{r,\Pi[a,b]} + \omega^{(0)}_{a,b}. \tag{2.9}$$

Proof: For any pair $t_k < t_l \in \Pi$ we have

$$\begin{aligned}
\|y_{t_k} - y_{t_l}\| &= \Big\|\sum_{j=k}^{l-1} y_{t_j,t_{j+1}}\Big\| = \Big\|\sum_{j=k}^{l-1} \big(F_{t_j,t_{j+1}} + \epsilon_{t_j,t_{j+1}}\big)\Big\| \\
&\le \Big\|\sum_{j=k}^{l-1} F_{t_j,t_{j+1}}\Big\| + \Big\|\sum_{j=k}^{l-1} \epsilon_{t_j,t_{j+1}}\Big\| \\
&\le \|I_{k,l}\| + \|F_{t_k,t_l}\| + \omega^{(0)}_{t_k,t_l}.
\end{aligned}$$

Therefore,

$$\|y_{t_k} - y_{t_l}\| \le \theta\omega^{\lambda}_{t_k,t_l} + \|F_{t_k,t_l}\| + \omega^{(0)}_{t_k,t_l},$$

which proves (2.9). □

2.4 Discrete greedy sequence of times

The original idea of a greedy sequence of times was introduced in [4, Definition 4.7] for the continuous time scale. In this paper, given a finite sequence of controls $\omega_{\cdot,\cdot} \in \mathcal{S}$ associated with parameters $\beta_\omega \in (0,1]$, we would like to construct a version of greedy sequence of times $\{\tau_m\}$ for the discrete time scale.
Let $[a,b] \subset \mathbb{R}$ be a closed interval of $\mathbb{R}$, and $\Pi[a,b] = \{t_i : 0 \leq i \leq n, a = t_0 < t_1 < \ldots < t_n = b\}$ be an arbitrary finite partition of $[a,b]$. Given a fixed $\gamma > 0$, assign the starting time $\tau_0 = a$. For each $m \in \mathbb{N}$, assume $\tau_m = t_k$ is determined. Then one can define τ_{m+1} by the following rule:

- if $\sum_{\omega\in\mathcal{S}} \omega^{\beta_\omega}_{t_k,t_{k+1}} > \gamma$ then set $\tau_{m+1} := t_{k+1}$;
- else set $\tau_{m+1} := \sup\{t_l \in (t_k, b] : \sum_{\omega\in\mathcal{S}} \omega^{\beta_\omega}_{t_k,t_l} \leq \gamma\}$.

Define $N(\gamma,[a,b])$ to be the number of times τ_m in $[a,b]$. From the definition,

$$\sum_{\omega\in\mathcal{S}} \omega^{\beta_\omega}_{\tau_m,\tau_{m+2}} > \gamma.$$

By taking both sides to the power of $\frac{1}{\beta}$ where $\beta := \min\{\beta_\omega : \omega \in \mathcal{S}\} \leq 1$, and using the Jensen's inequality, one obtains

$$\gamma^{\frac{1}{\beta}} < \Big(\sum_{\omega\in\mathcal{S}} \omega^{\beta_\omega}_{\tau_m,\tau_{m+2}}\Big)^{\frac{1}{\beta}} \leq (|\mathcal{S}|+1)^{\frac{1}{\beta}-1}\Big(\sum_{\omega\in\mathcal{S}} \omega^{\frac{\beta_\omega}{\beta}}_{\tau_m,\tau_{m+2}}\Big). \tag{2.10}$$

As a result, taking the sum of both sides of (2.10) as m going from 0 to $N(\gamma,[a,b])-3$ (whenever $N(\gamma,[a,b]) \geq 3$) and using the fact that all elements in the square brackets of (2.10) are controls, one obtains

$$\begin{aligned}[N(\gamma,[a,b])-2]\gamma^{\frac{1}{\beta}} &< \sum_{m=0}^{N(\gamma,[a,b])-3} (|\mathcal{S}|+1)^{\frac{1}{\beta}-1}\Big(\sum_{\omega\in\mathcal{S}} \omega^{\frac{\beta_\omega}{\beta}}_{\tau_m,\tau_{m+2}}\Big)\\ &< (|\mathcal{S}|+1)^{\frac{1}{\beta}-1}\sum_{\omega\in\mathcal{S}} 2\omega^{\frac{\beta_\omega}{\beta}}_{a,b}.\end{aligned}$$

Hence, $N(\gamma,[a,b])$ can be estimated as

$$N(\gamma,[a,b]) < 2 + \frac{2}{\gamma^{\frac{1}{\beta}}}(|\mathcal{S}|+1)^{\frac{1}{\beta}-1}\sum_{\omega\in\mathcal{S}} \omega^{\frac{\beta_\omega}{\beta}}_{a,b}. \tag{2.11}$$

Example 2.5

- For Young equations, one considers $\omega^{(1)}_{s,t} = C_f(t-s)$ with $\beta_1 = 1$ and $\omega^{(2)}_{s,t} = L_g^p \|x\|^p_{p,\Pi[s,t]}$ with $\beta_2 = \frac{1}{p}$ and $\mathcal{S} = \{\omega^{(1)},\omega^{(2)}\}$. Then $\beta = \frac{1}{p}$ and (2.11) has the form

$$N(\gamma,[a,b]) < 2 + \frac{2}{\gamma^p}3^{p-1}\Big[C_f^p(b-a)^p + L_g^p\|x\|^p_{p,\Pi[a,b]}\Big]. \tag{2.12}$$

- For rough equations, one consider $\mathcal{S} = \{\omega^{(1)},\omega^{(2)},\omega^{(3)}\}$, $\omega^{(1)}_{s,t} = t-s, \beta_1 = 1$, $\omega^{(2)}_{s,t} = L_g^p\|x\|^p_{p,\Pi[s,t]}, \beta_2 = \frac{1}{p}$ and $\omega^{(3)}_{s,t} = L_g^q\|\mathbb{X}\|^q_{q,\Pi[s,t]}, \beta_3 = \frac{1}{q} = \frac{2}{p}$. Then $\beta = \frac{1}{p}$ and (2.11) has the form

$$N(\gamma,[a,b]) < 2 + \frac{2}{\gamma^p}4^{p-1}\Big[C_f^p(b-a)^p + L_g^p\|x\|^p_{p,\Pi[a,b]} + L_g^q\|\mathbb{X}\|^q_{q,\Pi[a,b]}\Big]. \tag{2.13}$$

3 Discrete rough systems and solution estimates

We consider the discrete system (1.2) on Π driven by a discrete rough path $\mathbf{x}$. Our first observation is that F satisfies the assumption of the discrete sewing lemma 2.3, namely:

- If $F_{s,t} = g(y_s)x_{s,t}$ then by assigning $L_g := C_g$ one obtains

$$\|\delta F_{s,u,t}\| \leq C_g \|y_{s,u}\| \|x_{u,t}\| \leq L_g \left\| y \right\|_{p,\Pi[s,t]} \left\| x \right\|_{p,\Pi[s,t]}. \tag{3.1}$$

- If $F_{s,t} = g(y_s)x_{s,t} + Dg(y_s)g(y_s)\mathbb{X}_{s,t}$ where $g \in C_b^2$ or $g(y) = Cy + g(0)$ then a direct computation shows that

$$\begin{aligned}(\delta F)_{s,u,t} &= \left[-\int_0^1 Dg(y_s + \eta y_{s,u})(g(y_s)x_{s,u} + R^y_{s,u})d\eta \right] x_{u,t} + Dg(y_s)g(y_s)x_{s,u} \otimes x_{u,t} \\ &\quad + [Dg(y_s)g(y_s) - Dg(y_u)g(y_u)]\mathbb{X}_{u,t}.\end{aligned}$$

From the assumption $(\mathbf{H}_g)$, $Dg(y_\cdot)g(y_\cdot)$ is globally Lipschitz continuous with constant L_g^2 given by

$$L_g = \begin{cases} \sqrt{C_g^2 + C_g \|g\|_\infty} & \text{if} \quad g \in C_b^2; \\ C_g & \text{if} \quad g(y) = Cy + g(0). \end{cases}$$

Define

$$R^y_{s,t} := y_{s,t} - g(y_s)x_{s,t}, \ s,t \in \Pi.$$

As a result, it is easy to check that for a generic constant $K > 0$

$$\begin{aligned}\|(\delta F)_{s,u,t}\| &\leq L_g \|R^y_{s,u}\| \|x_{u,t}\| + L_g^2 \|y_{s,u}\| (\|x_{s,u} \otimes x_{u,t}\| + \|\mathbb{X}_{u,t}\|) \\ &\leq L_g \|R^y_{s,u}\| \|x_{u,t}\| + L_g^2 \|y_{s,u}\| (\|\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t}\| + \|\mathbb{X}_{u,t}\|) \\ &\leq K(L_g \left\| R^y \right\|_{q,\Pi[s,t]} \left\| x \right\|_{p,\Pi[s,t]} + L_g^2 \left\| y \right\|_{p,\Pi[s,t]} \left\| \mathbb{X} \right\|_{q,\Pi[s,t]}).\end{aligned} \tag{3.2}$$

Remark 3.1 It is easy to check that $\left(\left\| y \right\|_{p,\Pi[s,t]} \left\| x \right\|_{p,\Pi[s,t]} \right)^{\frac{p}{2}}$ for $1 \leq p < 2$, and $(\left\| R^y \right\|_{q,\Pi[s,t]} \left\| x \right\|_{p,\Pi[s,t]})^{\frac{p}{3}}$ and $(\left\| y \right\|_{p,\Pi[s,t]} \left\| \mathbb{X} \right\|_{q,\Pi[s,t]})^{\frac{p}{3}}$ for $2 \leq p < 3$, are control functions. Hence from now on, we can write symbolically $\omega^{\lambda_F}_{s,t}(\delta F)$ to indicate the right hand side of (3.1) or (3.2), where $\lambda_F = \frac{2}{p}$ for $1 < p < 2$ and $\lambda_F = \frac{3}{p}$ for $2 \leq p < 3$.

Introduce the norm

$$\|(y, R^y)\|_{p,\Pi[s,t]} := \|y_s\| + \left\| y \right\|_{p,\Pi[s,t]} + \left\| R^y \right\|_{q,\Pi[s,t]}, \ s \leq t.$$

One needs a technical estimates for p-variation norms.

Lemma 3.2 *Let $\{\tau_i\}_{i=\overline{0,m}} \subset \Pi$ be an sub partition of $[a,b]$. The following estimates hold*

$$\begin{aligned}\left\| y \right\|_{p,\Pi} &\leq m^{\frac{p-1}{p}} \sum_{k=0}^{m-1} \left\| y \right\|_{p,\Pi[\tau_k,\tau_{k+1}]}; \\ \left\| R^y \right\|_{q,\Pi} &\leq (2m)^{\frac{q-1}{q}} \Big(\sum_{k=0}^{m-1} \left\| R^y \right\|_{q,\Pi[\tau_k,\tau_{k+1}]} + C_g m^{\frac{1}{p}} \left\| x \right\|_{p,\Pi} \sum_{k=0}^{m-1} \left\| y \right\|_{p,\Pi[\tau_k,\tau_{k+1}]} \Big).\end{aligned} \tag{3.3}$$

Proof: The first estimate is simpler, thus it is enough to prove the second estimate. Observe that

$$\|(\delta R^y)_{s,u,t}\| \leq C_g \|y_{s,u}\| \|x_{u,t}\| \Rightarrow \|R^y_{s,t}\| \leq \|R^y_{s,u}\| + \|R^y_{u,t}\| + C_g \|y_{s,u}\| \|x_{u,t}\|, \quad \forall s \leq u \leq t. \quad (3.4)$$

Now given any finite partition $\Pi^* \subset \Pi$ of $[a,b]$, there exists a subsequence $\tau_l \leq \cdots \leq \tau_L$ of $\{\tau_i\}_{i=0}^m$ in the interval $[s,t]$ for every consecutive points s,t of Π^*. Then one can apply (3.4) to obtain

$$\|R^y_{s,t}\| \leq \|R^y_{s,\tau_l}\| + \|R^y_{\tau_l,\tau_{l+1}}\| + \ldots + \|R^y_{\tau_L,t}\| + C_g \left\|x\right\|_{p,\Pi[s,t]} \Big(\|y_{s,\tau_l}\| + \ldots + \|y_{\tau_L,t}\| \Big).$$

As a result, one can use the fact that $\|R^y\|^q_{q,\Pi[s,t]}$ and $\|y\|^p_{p,\Pi[s,t]}$ are controls and apply Jensen's inequality to obtain

$$\begin{aligned}
\Big(\sum_{s,t \in \Pi^*} \|R^y_{s,t}\|^q \Big)^{\frac{1}{q}} &\leq (2m)^{\frac{q-1}{q}} \Big\{ \Big(\sum_{s,t \in \Pi^*} (\|R^y_{s,\tau_l}\|^q + \|R^y_{\tau_l,\tau_{l+1}}\|^q + \ldots + \|R^y_{\tau_L,t}\|^q) \Big)^{\frac{1}{q}} \\
&\quad + C_g \Big(\sum_{s,t \in \Pi^*} \left\|x\right\|^q_{p,\Pi[s,t]} (\|y_{s,\tau_l}\|^q + \ldots + \|y_{\tau_L,t}\|^q) \Big)^{\frac{1}{q}} \Big\} \\
&\leq (2m)^{\frac{q-1}{q}} \Big\{ \Big(\sum_{s,t \in \Pi^*} (\|R^y_{s,\tau_l}\|^q + \|R^y_{\tau_l,\tau_{l+1}}\|^q + \ldots + \|R^y_{\tau_L,t}\|^q) \Big)^{\frac{1}{q}} \\
&\quad + C_g \Big(\sum_{s,t \in \Pi^*} m \left\|x\right\|^p_{p,\Pi[s,t]} \Big)^{\frac{1}{p}} \Big(\sum_{s,t \in \Pi^*} (\|y_{s,\tau_l}\|^p + \ldots + \|y_{\tau_L,t}\|^p) \Big)^{\frac{1}{p}} \Big\} \\
&\leq (2m)^{\frac{q-1}{q}} \Big\{ \Big(\sum_{i=0}^{m-1} \left\|R^y\right\|^q_{q,\Pi[\tau_i,\tau_{i+1}]}) \Big)^{\frac{1}{q}} + C_g m^{\frac{1}{p}} \left\|x\right\|_{p,\Pi} \Big(\sum_{i=0}^{m-1} \left\|y\right\|^p_{\tau_i,\tau_{i+1}} \Big)^{\frac{1}{p}} \Big\} \\
&\leq (2m)^{\frac{q-1}{q}} \Big\{ \sum_{k=0}^{m-1} \left\|R^y\right\|_{q,\Pi[\tau_k,\tau_{k+1}]} + C_g m^{\frac{1}{p}} \left\|x\right\|_{p,\Pi} \sum_{k=0}^{m-1} \left\|y\right\|_{p,\Pi[\tau_k,\tau_{k+1}]} \Big\}.
\end{aligned}$$

□

Our first main result in this section is formulated as follows.

Theorem 3.3 *There exist polynomials $\xi_i(L_g \left\|\mathbf{x}\right\|_{p,\Pi[a,b]}), i = 1,2,3$ of $L_g \left\|\mathbf{x}\right\|_{p,\Pi[a,b]}$ such that the solution of the rough difference equation* (1.2) *satisfies*

$$\begin{aligned}
&(i), \|y\|_{\infty,\Pi[a,b]} \leq \|y_a\| e^{\xi_1(L_g \left\|\mathbf{x}\right\|_{p,\Pi[a,b]})} + C_0 e^{\xi_2(L_g \left\|\mathbf{x}\right\|_{p,\Pi[a,b]})} - C_0; \\
&(ii), \|(y,R^y)\|_{p,\Pi[a,b]} \leq (\|y_a\| + C_0) e^{\xi_3(L_g \left\|\mathbf{x}\right\|_{p,\Pi[a,b]})}.
\end{aligned} \quad (3.5)$$

In case g is bounded, ξ_1 can be chosen independently of $\mathbf{x}$.

Proof: Starting from (1.2) for two consecutive moments t_k, t_{k+1} it follows from the discrete sewing lemma that

$$\begin{aligned}
\|y_{s,t} - F_{s,t}\| &\leq \| \sum_{t_k,t_{k+1} \in \Pi[s,t]} y_{t_k,t_{k+1}} - F_{s,t}\| \\
&\leq \sum_{t_k,t_{k+1} \in \Pi[s,t]} \|f(y_{t_k})\|(t_{k+1} - t_k) + \| \sum_{t_k,t_{k+1} \in \Pi[s,t]} F_{t_k,t_{k+1}} - F_{s,t}\| \\
&\leq \|f(y)\|_{\infty,\Pi[s,t]}(t-s) + K \omega^{\lambda_F}_{s,t}(\delta F),
\end{aligned}$$

where $\omega^{\lambda_F}_{s,t}(\delta F)$ is mentioned in Remark 3.1. As a result,

$$\left\|\!\left| y - F \right|\!\right\|_{p,\Pi[s,t]} \leq \|f(y)\|_{\infty,\Pi[s,t]}(t-s) + K\omega^{\lambda_F}_{s,t}(\delta F). \tag{3.6}$$

First, we consider the Young difference equation, i.e. $F_{s,t} = g(y_s)x_{s,t}$ and $L_g = C_g$. It then follows from (3.1) and (3.6) that there exists generic constants K and C_0 such that

$$\begin{aligned}
\left\|\!\left| y \right|\!\right\|_{p,\Pi[s,t]} &\leq \|f(y)\|_{\infty,\Pi[s,t]}(t-s) + \|g(y)\|_{\infty,\Pi[s,t]} \left\|\!\left| x \right|\!\right\|_{p,\Pi[s,t]} + KL_g \left\|\!\left| y \right|\!\right\|_{p,\Pi[s,t]} \left\|\!\left| x \right|\!\right\|_{p,\Pi[s,t]} \\
&\leq (C_f\|y\|_{\infty,\Pi[s,t]} + \|f(0)\|)(t-s) \\
&\quad +(L_g\|y\|_{\infty,\Pi[s,t]} + \|g(0)\|) \left\|\!\left| x \right|\!\right\|_{p,\Pi[s,t]} + KL_g \left\|\!\left| y \right|\!\right\|_{p,\Pi[s,t]} \left\|\!\left| x \right|\!\right\|_{p,\Pi[s,t]} \\
&\leq K\Big[C_f(t-s) + L_g \left\|\!\left| x \right|\!\right\|_{p,\Pi[s,t]}\Big](\|y\|_{p,\Pi[s,t]} + C_0).
\end{aligned} \tag{3.7}$$

If $C_f(t-s) + L_g \left\|\!\left| x \right|\!\right\|_{p,\Pi[s,t]} \leq \frac{1}{2K}$ then by taking the term containing $\left\|\!\left| y \right|\!\right\|_{p,\Pi[s,t]}$ to the left hand side, it follows from (3.7) that

$$\left\|\!\left| y \right|\!\right\|_{p,\Pi[s,t]} \leq K\Big[C_f(t-s) + L_g \left\|\!\left| x \right|\!\right\|_{p,\Pi[s,t]}\Big][\|y_s\| + C_0]$$

and then

$$\begin{aligned}
\|y\|_{p,\Pi[s,t]} + C_0 \leq \|y_s\| + C_0 + \left\|\!\left| y \right|\!\right\|_{p,\Pi[s,t]} &\leq \Big\{1 + K\Big[C_f(t-s) + L_g \left\|\!\left| x \right|\!\right\|_{p,\Pi[s,t]}\Big]\Big\}[\|y_s\| + C_0] \\
&\leq [\|y_s\| + C_0]e^{K\Big[C_f(t-s)+L_g\left\|\!\left| x \right|\!\right\|_{p,\Pi[s,t]}\Big]}.
\end{aligned} \tag{3.8}$$

This motivates us to construct, by using the construction in Section 2.4 for the controls $\omega^{(1)}_{s,t} = C_f(t-s)$ with $\beta_1 = 1$ and $\omega^{(2)}_{s,t} = L_g^p \left\|\!\left| x \right|\!\right\|^p_{p,\Pi[s,t]}$ with $\beta_2 = \frac{1}{p}$, the greedy sequence of times $\tau_m = \{\tau_m(\gamma,[a,b])\}$ for the discrete time set Π on $[a,b]$ where γ is some positive constant less than $\frac{1}{2K}$. To estimate the solution norm, observe that on each $[\tau_m,\tau_{m+1}]$ such that $C_f(\tau_{m+1}-\tau_m) + L_g \left\|\!\left| x \right|\!\right\|_{p,\Pi[\tau_m,\tau_{m+1}]} \leq \frac{1}{2K}$, one obtains

$$\begin{aligned}
\|y_{\tau_{m+1}}\| + C_0 &\leq \|y\|_{\infty,\Pi[\tau_m,\tau_{m+1}]} + C_0 \\
&\leq \|y\|_{p,\Pi[\tau_m,\tau_{m+1}]} + C_0 \\
&\leq \exp\Big\{K\Big[C_f(\tau_{m+1}-\tau_m) + L_g \left\|\!\left| x \right|\!\right\|_{p,\Pi[\tau_m,\tau_{m+1}]}\Big]\Big\}(\|y_{\tau_m}\| + C_0).
\end{aligned} \tag{3.9}$$

Meanwhile, if $C_f(\tau_{m+1}-\tau_m) + L_g \left\|\!\left| x \right|\!\right\|_{p,\Pi[\tau_m,\tau_{m+1}]}) > \gamma$ (hence τ_m, τ_{m+1} are two consecutive discrete times in Π) then it follows from (1.2) that

$$\begin{aligned}
\|y_{\tau_{m+1}}\| + C_0 &\leq C_0 + \|y_{\tau_m}\| + \|y_{\tau_m}\|\Big[C_f(\tau_{m+1}-\tau_m) + L_g\|x_{\tau_m,\tau_{m+1}}\|\Big] \\
&\quad +C_0\Big[C_f(\tau_{m+1}-\tau_m) + L_g\|x_{\tau_m,\tau_{m+1}}\|\Big] \\
&\leq \Big(1 + C_f(\tau_{m+1}-\tau_m) + L_g\|x_{\tau_m,\tau_{m+1}}\|\Big)(\|y_{\tau_m}\| + C_0) \\
&\leq \exp\{C_f(\tau_{m+1}-\tau_m) + L_g\|x_{\tau_m,\tau_{m+1}}\|\}(\|y_{\tau_m}\| + C_0)
\end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10) yields, for all $m = 0,\ldots,N(\gamma,[a,b])-1$,

$$\begin{aligned}
\|y\|_{p,\Pi[\tau_m,\tau_{m+1}]} + C_0 &\leq \exp\Big\{K\Big[C_f(\tau_{m+1}-\tau_m) + L_g \left\|\!\left| x \right|\!\right\|_{p,\Pi[\tau_m,\tau_{m+1}]}\Big]\Big\}(\|y_{\tau_m}\| + C_0) \\
&\leq \exp\Big\{K\Big[C_f(\tau_{m+1}-\tau_m) + L_g^p \left\|\!\left| x \right|\!\right\|^p_{p,\Pi[\tau_m,\tau_{m+1}]} + 1\Big]\Big\}(\|y_{\tau_m}\| + C_0).
\end{aligned} \tag{3.11}$$

Note that $\|x\|^p_{p,\Pi[s,t]}$ is a control, hence one can prove by induction from (3.11) that

$$
\begin{aligned}
\|y\|_{\infty,\Pi[a,b]} + C_0 &\leq \exp\Big\{K\Big[C_f(b-a) + L_g^p \left\|x\right\|^p_{p,\Pi[a,b]} + N(\gamma,[a,b])\Big]\Big\}(\|y_a\| + C_0),\\
\|y\|_{p,\Pi[a,b]} + C_0 &\leq \|y_a\| + C_0 + N(\gamma,[a,b])^{\frac{p-1}{p}} \sum_{m=0}^{N(\gamma,[a,b])-1} \left\|y\right\|_{p,\Pi[\tau_m,\tau_{m+1}]}\\
&\leq N(\gamma,[a,b])^{\frac{2p-1}{p}} \times \qquad (3.12)\\
&\quad \times \exp\Big\{K\Big[C_f(b-a) + L_g^p \left\|x\right\|^p_{p,\Pi[a,b]} + N(\gamma,[a,b])\Big]\Big\}(\|y_a\| + C_0),
\end{aligned}
$$

which, together with (2.12), proves (3.5).

Next, consider the rough difference equation i.e. $F_{s,t} = g(y_s)x_{s,t} + Dg(y_s)g(y_s)\mathbb{X}_{s,t}$ then it follows from (3.2) and (3.6) that for generic constants $K > 1, C_0 > 0$

$$
\begin{aligned}
\left\|y\right\|_{p,\Pi[s,t]} &\leq (C_f\|y\|_{\infty,\Pi[s,t]} + C_0)(t-s) + (L_g\|y\|_{\infty,\Pi[s,t]} + C_0)(\left\|x\right\|_{p,\Pi[s,t]} + L_g \left\|\mathbb{X}\right\|_{q,\Pi[s,t]})\\
&\quad + K \left\|(y,R^y)\right\|_{p,\Pi[s,t]} (L_g \left\|x\right\|_{p,\Pi[s,t]} + L_g^2 \left\|\mathbb{X}\right\|_{q,\Pi[s,t]})\\
\left\|R^y\right\|_{p,\Pi[s,t]} &\leq (C_f\|y\|_{\infty,\Pi[s,t]} + C_0)(t-s) + (L_g\|y\|_{\infty,\Pi[s,t]} + C_0)L_g \left\|\mathbb{X}\right\|_{q,\Pi[s,t]}\\
&\quad + K \left\|(y,R^y)\right\|_{p,\Pi[s,t]} (L_g \left\|x\right\|_{p,\Pi[s,t]} + L_g^2 \left\|\mathbb{X}\right\|_{q,\Pi[s,t]}).
\end{aligned}
$$

it follows from the assumptions of f and g that

$$
\|(y,R^y)\|_{p,[s,t]} + C_0 \leq \|y_s\| + C_0 + K\Big[C_f(t-s) + L_g \left\|x\right\|_{p,\Pi[s,t]} + L_g^2 \left\|\mathbb{X}\right\|_{q,\Pi[s,t]}\Big](\|(y,R^y)\|_{p,\Pi[s,t]} + C_0). \qquad (3.13)
$$

That motivates to construct the sequence of times $\{\tau_m(\gamma,[a,b])\}$, for $\gamma \leq \frac{1}{2K}$, $\mathcal{S} = \{\omega^{(1)},\omega^{(2)},\omega^{(3)}\}$, $\omega^{(1)}_{s,t} = t-s, \beta_1 = 1$, $\omega^{(2)}_{s,t} = L_g^p \left\|x\right\|^p_{p,\Pi[s,t]}, \beta_2 = \frac{1}{p}$ and $\omega^{(3)}_{s,t} = L_g^p \left\|\mathbb{X}\right\|^q_{q,\Pi[s,t]}, \beta_3 = \frac{1}{q}$. The arguments are now similar to the rest of the Young case, with note of inequalities (2.13) and (3.3).

Finally, in case g is bounded, one can neglect L_g in (3.9) and the first line of the estimate (3.10) and repeat the above argument to obtain similar estimate in which ξ_1 just depends on $b-a$. □

4 Discrete random dynamical systems and attractors

4.1 Discrete rough cocyles and random dynamical systems

The generation of random dynamical systems from rough systems is studied in [2] and [7] for the continuous time, and in [10] for the discrete time. In this section, we rephrase the construction for a general discrete time setting. Given $p \in (0,3)$ and a regular discrete time set $\Pi^\Delta = \{k\Delta : k \in \mathbb{Z}\}$, denote by $\mathcal{C}^p_0(\Pi^\Delta, T^2_1(\mathbb{R}^m))$ the space of all paths $\mathbf{g} : \Pi^\Delta \to T^2_1(\mathbb{R}^m))$. Then $\mathcal{C}^p_0(\Pi^\Delta, T^2_1(\mathbb{R}^m))$ is equipped with the compact open topology given by the $p-$ variation norm (2.2), i.e the topology generated by the metric

$$
d^\Delta_p(\mathbf{g},\mathbf{h}) := \sum_{k\geq 1} \frac{1}{2^k}(\|\mathbf{g}-\mathbf{h}\|_{p,\Pi[-k\Delta,k\Delta]} \wedge 1).
$$

As a result, it is separable and thus a Polish space.

Let us consider a stochastic process $\bar{\mathbf{X}}$ defined on a probability space $(\bar\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with realizations in $(\mathcal{C}^p_0(\Pi^\Delta, T^2_1(\mathbb{R}^m)), \mathcal{F})$. Assume further that $\bar{\mathbf{X}}$ has stationary increments. Assign $\Omega :=$

$\mathcal{C}_0^p(\Pi^\Delta, T_1^2(\mathbb{R}^m))$ and equip it with the Borel σ - algebra $\mathcal{F}$ and let $\mathbb{P}$ be the law of $\bar{\mathbf{X}}$. Denote by θ the *Wiener-type shift*

$$(\theta_t\omega)_\cdot = \omega_t^{-1} \bullet \omega_{t+\cdot}, \quad \forall t \in \Pi^\Delta, \omega \in \mathcal{C}_0^p(\Pi^\Delta, T_1^2(\mathbb{R}^m)), \tag{4.1}$$

and define the so-called *diagonal process* $\mathbf{X} : \Pi^\Delta \times \Omega \to T_1^2(\mathbb{R}^m), \mathbf{X}_t(\omega) = \omega_t$ for all $t \in \Pi^\Delta, \omega \in \Omega$. Due to the stationarity of $\bar{\mathbf{X}}$, it can be proved that θ is invariant under $\mathbb{P}$, then forming a discrete (and thus measurable) dynamical system on $(\Omega, \mathcal{F}, \mathbb{P})$ [2, Theorem 5]. Moreover, $\mathbf{X}$ forms an $p-$ *rough path cocycle*, namely, $\mathbf{X}_\cdot(\omega) \in \mathcal{C}_0^p(\Pi^\Delta, T_1^2(\mathbb{R}^m))$ for every $\omega \in \Omega$, which satisfies the *cocyle relation*:

$$\mathbf{X}_{t+s}(\omega) = \mathbf{X}_s(\omega) \bullet \mathbf{X}_t(\theta_s\omega), \forall \omega \in \Omega, t, s \in \Pi^\Delta,$$

in the sense that $\mathbf{X}_{s,s+t} = \mathbf{X}_t(\theta_s\omega)$ with the increment notation $\mathbf{X}_{s,s+t} := \mathbf{X}_s^{-1} \bullet \mathbf{X}_{s+t}$. In particular, the Wiener shift (4.1) implies that

$$\left\| \mathbf{x}(\theta_h\omega) \right\|_{p,\Pi[s,t]} = \left\| \mathbf{x}(\omega) \right\|_{p,\Pi[s+h,t+h]}, \quad \forall s, t, h \in \Pi^\Delta. \tag{4.2}$$

Define the discrete mapping $H^\Delta(\mathbf{x})y = y + f(y)\Delta + F_{0,\Delta}(y, \mathbf{x})$. Similar to [10], we can easily prove that the discrete Euler scheme (1.2) generates a discrete random dynamical system φ^Δ : $\Pi^\Delta \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ such that

$$\varphi^\Delta(0, \omega)y_0 = y_0, \quad \varphi^\Delta(k\Delta, \omega)y_0 := H^\Delta(\theta_{(k-1)h}\omega) \circ \ldots \circ H^\Delta(\omega)y_0, \quad \forall k \geq 1. \tag{4.3}$$

Throughout this paper, we assume that θ is an ergodic dynamical system.

4.2 Random attractors

Given a random dynamical system φ on the phase space $\mathbb{R}^d$, we follow [1, Chapter 9] and the references therein) to present the notion of random pullback attractors. Roughly speaking, an invariant random compact set $\mathcal{A} \in \mathcal{D}$ is called a *pullback attractor* in $\mathcal{D}$, if $\mathcal{A}$ attracts any closed random set $\hat{D} \in \mathcal{D}$ in the pullback sense, i.e.

$$\lim_{t\to\infty} d_H(\varphi(t, \theta_{-t}\omega)\hat{D}(\theta_{-t}\omega)|\mathcal{A}(\omega)) = 0, \tag{4.4}$$

where $d_H(\cdot|\cdot)$ is the Hausdorff semi-distance, i.e. $d_H(D|A) := \sup_{d\in D} \inf_{a\in A} \|d - a\|$. The existence of a pullback attractor follows from the existence of a pullback absorbing set (see [10] and the references therein), namely a random set $\mathcal{B} \in \mathcal{D}$ is called *pullback absorbing* in the universe $\mathcal{D}$ if $\mathcal{B}$ absorbs all closed random sets in $\mathcal{D}$, i.e. for any closed random set $\hat{D} \in \mathcal{D}$, there exists a time $t_0 = t_0(\omega, \hat{D})$ such that

$$\varphi(t, \theta_{-t}\omega)\hat{D}(\theta_{-t}\omega) \subset \mathcal{B}(\omega), \text{ for all } t \geq t_0. \tag{4.5}$$

Then given the universe $\mathcal{D}$ and a compact pullback absorbing set $\mathcal{B} \in \mathcal{D}$, there exists a unique pullback attractor $\mathcal{A}(\omega)$ in $\mathcal{D}$, given by

$$\mathcal{A}(\omega) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} \varphi(s, \theta_{-s}\omega)\mathcal{B}(\theta_{-s}\omega)}. \tag{4.6}$$

We quote the following result from [5, Lemma 5.2].

Lemma 4.1 *(i) Let $a : \Omega \to [0,\infty)$ be a random variable, $\log(1 + a(\cdot)) \in L^1$ and $\hat{a} := E\log(1 + a(\cdot)) = \int_\Omega \log(1 + a(\cdot))d\mathbb{P}$. Let $\lambda > \hat{a}$ be an arbitrary fixed positive number. Put*

$$b(x) := \sum_{k=1}^{\infty} e^{-\lambda k} \prod_{i=0}^{k-1} (1 + a(\theta_{-i}x)).$$

Then $b(\cdot)$ is a nonnegative almost everywhere finite and tempered random variable.
(ii) Let $c : \Omega \to [0,\infty)$ be a tempered random variable, and $\delta > 0$ be an arbitrary fixed positive number. Put

$$d(x) := \sum_{k=1}^{\infty} e^{-\delta k} c(\theta_{-k}x).$$

Then $d(\cdot)$ is a nonnegative almost everywhere finite and tempered random variable.

Now we assume further that f is dissipative: there exist $c, d > 0$ so that for all $y \in \mathbb{R}^d$,

$$\langle y, f(y)\rangle \leq c - d\|y\|^2. \tag{4.7}$$

In addition, assume the p-variation norm of $\mathbf{x}$ is of finite moment for all order $k \in \mathbb{N}$, i.e.

$$\mathbb{E}\, \|\mathbf{x}(\cdot)\|^k_{p,\Pi[a,b]} < \infty, \quad \forall k \in \mathbb{N}, \quad \forall 0 < a < b,\ a, b \in \Pi. \tag{4.8}$$

It is proved in [10] that the discrete system (1.2), with g bounded and for the regular step size Δ small enough, admits a discrete pullback attractor $\mathcal{A}^\Delta$. Thanks to the discrete sewing lemma, we can now prove a more general result for discrete system (1.2), which is formulated as follows.

Theorem 4.2 *Assume ($\mathbf{H_f}$), ($\mathbf{H_g}$) and the dissipativity condition* (4.7) *and condition* (4.8)*. Consider system* (1.2) *with the regular time set Π^Δ, where $\Delta \in (0,1)$ satisfies the inequality*

$$0 < \Delta < 1 \wedge \frac{d}{2C_f^2} \wedge \frac{1}{2d}. \tag{4.9}$$

Then for L_g small enough, the generated discrete random dynamical system φ^Δ of (1.2) *admits a random pullback attractor $\mathcal{A}^\Delta$.*

Proof: Given $\Delta \in (0,1)$ small enough, one introduces $n \in \mathbb{N}$ such that $T := n\Delta \leq 1 < (n+1)\Delta$. First, one applies the Lyapunov function $\|y\|^2$ into (1.2) and uses Cauchy-Schwarz inequality to obtain for $t_k = k\Delta,\ k = 0, \ldots, (n-1)$

$$\begin{aligned}
\|y_{t_{k+1}}\|^2 &= \|y_{t_k}\|^2 + 2\langle y_{t_k}, f(y_{t_k})\rangle\Delta + \|f(y_{t_k})\|^2\Delta^2 + 2\langle y_{t_k}, F_{t_k,t_{k+1}}\rangle \\
&\quad +2\langle f(y_{t_k}), F_{t_k,t_{k+1}}\rangle\Delta + \|F_{t_k,t_{k+1}}\|^2 \\
&\leq \|y_{t_k}\|^2 + 2(c - d\|y_{t_k}\|^2)\Delta + 2(C_f^2\|y_{t_k}\|^2 + \|f(0)\|^2)\Delta^2 + 2\langle y_{t_k}, F_{t_k,t_{k+1}}\rangle \\
&\quad +2\|f(y)\|_{\infty,\Pi[0,T]}\|F\|_{\infty,\Pi[0,T]}\Delta + \|F_{t_k,t_{k+1}}\|^2 \\
&\leq \|y_{t_k}\|^2(1 - 2d\Delta + 2C_f^2\Delta^2) + K(\Delta + \Delta^2) + 2\|f(y)\|_{\infty,\Pi[0,T]}\|F\|_{\infty,\Pi[0,T]}\Delta \\
&\quad +2\langle y_{t_k}, F_{t_k,t_{k+1}}\rangle + \|F_{t_k,t_{k+1}}\|^2.
\end{aligned} \tag{4.10}$$

It follows from (4.9) that $0 < 1 - 2d\Delta + 2C_f^2\Delta^2 < 1 - d\Delta < e^{-d\Delta}$. Then (4.10) leads to, for a generic constant $K > 1$, the following estimate

$$\|y_{t_{k+1}}\|^2 \leq e^{-d\Delta}\|y_{t_k}\|^2 + \left[2\langle y_{t_k}, F_{t_k,t_{k+1}}\rangle + \|F_{t_k,t_{k+1}}\|^2\right] + K\Delta(1 + \|f(y)\|_{\infty,\Pi[0,T]}\|F\|_{\infty,\Pi[0,T]}).$$

Assign $G_{s,t} := 2\langle y_s, F_{s,t}\rangle + \|F_{s,t}\|^2$. By induction one can prove that

$$\begin{aligned}
&\|y_{t_n}\|^2 \\
\leq\ & e^{-n\Delta d}\|y_0\|^2 + K\Delta\left(1+\|f(y)\|_{\infty,\Pi[0,T]}\|F\|_{\infty,\Pi[0,T]}\right)\left(\sum_{k=0}^{n-1} e^{-d(n-k)\Delta}\right) + \sum_{k=0}^{n-1} e^{-d(t_n-t_k)}G_{t_k,t_{k+1}} \\
\leq\ & e^{-dT}\|y_0\|^2 + \frac{1}{1-e^{-d\Delta}}K\Delta\left(1+\|f(y)\|_{\infty,\Pi[0,T]}\|F\|_{\infty,[0,T]}\right) + \sum_{k=0}^{n-1} e^{-d(t_n-t_k)}G_{t_k,t_{k+1}} \\
\leq\ & e^{-dT}\|y_0\|^2 + K\left(1+\|f(y)\|_{\infty,\Pi[0,T]}\|F\|_{\infty,\Pi[0,T]}\right) + \sum_{k=0}^{n-1} e^{-d(t_n-t_k)}G_{t_k,t_{k+1}}. \qquad (4.11)
\end{aligned}$$

Let us estimate the third term in the right hand side of (4.11). Define $\tilde{G}_{s,t} := e^{-d(t_n-s)}G_{s,t}$. Observe that

$$\begin{aligned}
\|(\delta\tilde{G})_{s,u,t}\| &\leq (e^{-d(t_n-s)} - e^{-d(t_n-u)})\|G_{u,t}\| + \|e^{-d(t_n-s)}\|\|(\delta G)_{s,u,t}\| \\
&\leq (t-s)\|G\|_{\infty,\Pi[s,t]} + \|(\delta G)_{s,u,t}\| \\
&\leq (t-s)\|F\|_{\infty,\Pi[s,t]}\Big(2\|y\|_{\infty,\Pi[0,T]} + \|F\|_{\infty,\Pi[0,T]}\Big) + \|(\delta G)_{s,u,t}\|,\ 0\leq s\leq u\leq t\leq T
\end{aligned}$$

where the first term has the form of a control w.r.t. a suitable $\lambda > 1$. The second term is estimated as follows

$$\begin{aligned}
\|(\delta G)_{s,u,t}\| &\leq \Big\|2\langle y_s, F_{s,t}-F_{s,u}\rangle - 2\langle y_u, F_{u,t}\rangle + 2\langle F_{s,u}, F_{u,t}\rangle + 2\langle F_{s,u}+F_{u,t}, (\delta F)_{s,u,t}\rangle + [(\delta F)_{s,u,t}]^2\Big\| \\
&\leq \Big\|2\langle y_s, \delta F_{s,u,t}\rangle - \langle y_{s,u}-F_{s,u}, F_{u,t}\rangle + \langle F_{s,t}+F_{s,u}+F_{u,t}, (\delta F)_{s,u,t}\rangle\Big\| \\
&\leq \Big\|2\langle y_s, (\delta F)_{s,u,t}\rangle - \langle y_{s,u}-F_{s,u}, F_{u,t}\rangle\Big\| + 3\|F\|_{\infty,\Pi[0,T]}\|(\delta F)_{s,u,t}\| \\
&\leq \|(\delta F)_{s,u,t}\|\Big(2\|y\|_{\infty,\Pi[0,T]} + 3\|F\|_{\infty,\Pi[0,T]}\Big) + 2\|y_{s,u}-F_{s,u}\|\|F\|_{\infty,\Pi[s,t]} \\
&\leq \omega^{\lambda_F}_{s,t}(\delta F)\Big(2\|y\|_{\infty,\Pi[0,T]} + 3\|F\|_{\infty,\Pi[0,T]}\Big) \\
&\quad +2\Big(\|f(y)\|_{\infty,\Pi[0,T]}(t-s) + K\omega^{\lambda_F}_{s,t}(\delta F)\Big)\|F\|_{\infty,\Pi[0,T]}
\end{aligned}$$

where the last inequality is due to the discrete sewing lemma to estimate $y_{s,u} - F_{s,u}$ (see also (3.6)). As a result, $\delta\tilde{G}_{s,u,t}$ is bounded by a control up to a power $\lambda_{\tilde{G}}$ with

$$\begin{aligned}
\omega^{\lambda_{\tilde{G}}}_{s,t}(\delta\tilde{G}) &= (t-s)\|F\|_{\infty,[s,t]}\Big(2\|y\|_{\infty,\Pi[0,T]} + \|F\|_{\infty,\Pi[0,T]}\Big) \\
&\quad +\omega^{\lambda_F}_{s,t}(\delta F)\Big(2\|y\|_{\infty,\Pi[0,T]} + 3\|F\|_{\infty,\Pi[0,T]}\Big) \\
&\quad +2\Big(\|f(y)\|_{\infty,\Pi[0,T]}(t-s) + K\omega^{\lambda_F}_{s,t}(\delta F)\Big)\|F\|_{\infty,[s,t]} \\
&\leq K(t-s)\|F\|_{\infty,[s,t]}\Big(\|y\|_{\infty,\Pi[0,T]} + \|F\|_{\infty,\Pi[0,T]} + 1\Big) \\
&\quad +K\omega^{\lambda_F}_{s,t}(\delta F)\Big(\|y\|_{\infty,\Pi[0,T]} + \|F\|_{\infty,\Pi[0,T]}\Big). \qquad (4.12)
\end{aligned}$$

Therefore, by applying the discrete sewing lemma to (4.11), one obtains, for a generic constant K

$$\|y_T\|^2 \leq e^{-dT}\|y_0\|^2 + K\left(1+\|f(y)\|_{\infty,\Pi[0,T]}\|F\|_{\infty,\Pi[0,T]}\right) + e^{-dT}G_{0,T} + K\omega^{\lambda_{\tilde{G}}}_{0,T}(\delta\tilde{G})$$

$$\begin{aligned}&\leq e^{-dT}\|y_0\|^2 + K\Big[1+(\|y\|_{\infty,\Pi[0,T]}+C_0)\|F\|_{\infty,\Pi[0,T]}+\|F\|^2_{\infty,\Pi[0,T]}\Big]\\&\quad +K\omega^{\lambda_F}_{s,t}(\delta F)\Big(\|y\|_{\infty,\Pi[0,T]}+\|F\|_{\infty,\Pi[0,T]}\Big).\end{aligned} \tag{4.13}$$

Taking into account Remark 3.1 and by replacing $\omega^{\lambda_F}_{s,t}(\delta F)$ by the right hand side of (3.1) or (3.2), then using the estimates (3.5) in Theorem 3.3, and the estimate

$$\|F\|_{\infty,\Pi[s,t]} \leq (L_g\|y\|_{\infty,\Pi[s,t]}+C_0)\,\|\mathbf{x}\|_{p,\Pi[s,t]}$$

one concludes that there exists generic constants $C_0 > 0$, a generic polynomial ξ_1, and a random variable ξ_2 of the form $\xi_2(z) = z^m, z \in \mathbb{R}^+$ for some (fractional order) $m \geq p$ such that (4.12) has the form

$$\begin{aligned}\|y_T\|^2 &\leq e^{-dT}\Big(1+L_g\xi_1(\|\mathbf{x}\|_{p,\Pi[0,T]})e^{\xi_2(L_g\|\mathbf{x}\|_{p,\Pi[0,T]})}\Big)\|y_0\|^2\\&\quad +C_0\xi_1(\|\mathbf{x}\|_{p,\Pi[0,T]})e^{\xi_2(L_g\|\mathbf{x}\|_{p,\Pi[0,T]})}\end{aligned} \tag{4.14}$$

One now applies the inequality $\log(1+\mu e^{\nu}) \leq \mu+\nu, \forall \mu,\nu \in \mathbb{R}, \nu \geq 0$, to obtain

$$\log\Big(1+L_g\xi_1(\|\mathbf{x}\|_{p,\Pi[0,T]})e^{\xi_2(L_g\|\mathbf{x}\|_{p,\Pi[0,T]})}\Big) \leq L_g\xi_1(\|\mathbf{x}\|_{p,\Pi[0,T]})+\xi_2(L_g\,\|\mathbf{x}\|_{p,\Pi[0,T]}) \in L^1. \tag{4.15}$$

Hence, one can choose L_g small enough such that

$$L_g\mathbb{E}\xi_1(\|\mathbf{x}(\cdot)\|_{p,\Pi[0,T]})+\mathbb{E}\xi_2(L_g\,\|\mathbf{x}(\cdot)\|_{p,\Pi[0,T]}) < dT. \tag{4.16}$$

Similarly to [5], one can use (4.14), (4.15) together with Lemma 4.1 to prove that there exists an absorbing set of the form $B^\Delta(\omega) = \bar{B}(0,R^\Delta(\mathbf{x}(\omega)))$, where

$$R^\Delta(\mathbf{x}) = 1+C_0\sum_{k\geq 1} e^{-dTk}\Lambda(\mathbf{x},\Pi[-kT,-(k-1)T])\prod_{i=1}^{k}(1+L_g\Lambda(\mathbf{x},\Pi[-iT,-(i-1)T])) \tag{4.17}$$

with $T = n\Delta$ and $\Lambda(\mathbf{x},\Pi[a,b]) = \xi_1(\|\mathbf{x}\|_{p,\Pi[a,b]})e^{\xi_2(L_g\|\mathbf{x}\|_{p,\Pi[a,b]})}$. Due to (4.6), there exists a random pullback attractor $\mathcal{A}^\Delta(\omega)$ which is contained in $B^\Delta(\omega)$.

□

Remark 4.3 Note that condition (4.16) should be tested on the discrete time set Π^Δ, it is natural to think that the choice of L_g depends on Δ. However, it is not always the case. Indeed, in case $\mathbf{x}$ is induced on Π by a continuous stochastic process $X_t, t \in \mathbb{R}$ which has almost all realizations of ν-Hölder continuity for $\nu \in (\frac{1}{3},1)$. Then each realization x can be lifted to a continuous rough path $\mathbf{x}$, and the probability space can be constructed by using arguments in [2] (see also [7] or [10]). In that case, by fixing a positive number $\Delta_0 < 1 \wedge \frac{d}{2C_f^2} \wedge \frac{1}{2d}$, condition (4.16) is satisfied if the following criterion holds

$$L_g\mathbb{E}\xi_1(\|\mathbf{x}(\cdot)\|_{p,[0,1]})+\mathbb{E}\xi_2(L_g\,\|\mathbf{x}(\cdot)\|_{p,[0,1]}) < d(1-\Delta_0) \tag{4.18}$$

for all $\Delta \in (0,\Delta_0)$ since $1 \geq T = n\Delta > 1-\Delta > 1-\Delta_0$. Condition (4.18) shows that L_g is actually chosen independently of Δ.

5 Semi-continuity of numerical attractors

In this section we consider the following stochastic differential equation

$$d\hat{y}_t = f(\hat{y}_t)dt + g(\hat{y}_t)dX_t, \hat{y}_a \in \mathbb{R}^d, t \in [a,b] \tag{5.1}$$

where f, g satisfy conditions ($\mathbf{H_f}$), ($\mathbf{H_g}$) and X is a stochastic process such that it can be lifted to a stationary increment process $(x_\cdot(\omega), \mathbb{X}_{\cdot,\cdot}(\omega))$ of which all realizations $\mathbf{x} = (x, \mathbb{X})$ are α- Hölder rough paths for $\alpha \in (1/3, 1/2]$ (examples of such processes can be found in [12, Chapter 10]). This allows us to write (5.1) in the pathwise sense, i.e. for each realized rough path $\mathbf{x}$

$$\hat{y}_t = \hat{y}_a + \int_a^t f(\hat{y}_s)ds + \int_a^t g(\hat{y}_s)d\mathbf{x}_s, \quad t \in [a,b]$$

in which the second integral is understood as rough integral (see [14]).

It is proved in [10] that there exists a unique solution $\hat{y}(t, \mathbf{x}, y_a), t \in [a,b]$ to (5.1). Moreover, one can use similar estimates to Theorem 3.3 to prove that: there exists a generic integrable random variable $\hat{\xi}(\mathbf{x})$ such that for $0 < a < b$

$$\begin{aligned}
\|\hat{y}\|_{\infty,[a,b]} &\leq (\|y_a\| + 1)\hat{\xi}(\mathbf{x}, [a,b]) \\
\|\hat{y}\|_{p,[a,b]} &\leq (\|y_a\| + 1)\hat{\xi}(\mathbf{x}, [a,b])(b - a + \|x\|_{p,[a,b]} + \|\mathbb{X}\|_{q,[a,b]}) \\
\left\|R^{\hat{y}}\right\|_{q,[a,b]} &\leq (\|y_a\| + 1)\hat{\xi}(\mathbf{x}, [a,b])(b - a + \|\mathbb{X}\|_{q,[a,b]}).
\end{aligned} \tag{5.2}$$

We define the numerical solution by the Euler scheme (1.2) with $y_a = \hat{y}_a$ and raise the question on the convergence of numerical solution defined in (1.2) to z. This problem is solved in [6], [15] for instant, also in [10] with the cut-off technique. Below we will present a direct proof using the discrete sewing Lemma 2.2.

Proposition 5.1 *If in additional to ($\mathbf{H_f}$) , ($\mathbf{H_g}$), f and D^2g are locally Lipchitz, then there exists a generic random variable $\hat{\xi}(\hat{y}_a, \mathbf{x}, [a,b])$ such that the following estimate holds*

$$\sup_{0\leq k\leq m} \|\hat{y}(t_k, y_a, x) - y_{t_k}\| \leq \hat{\xi}(\hat{y}_a, \mathbf{x}, [a,b])(|\Pi[a,b]| + \|\mathbf{x}\|_{p,[a,b],|\Pi[a,b]|})^{3-p}(b - a + \|\mathbf{x}\|^p_{p,[a,b]}). \tag{5.3}$$

In particular,

$$\lim_{|\Pi[a,b]|\to 0} \sup_{0\leq k\leq m} \|\hat{y}(t_k, \hat{y}_a, x) - y_{t_k}\| = 0. \tag{5.4}$$

Proof: We only prove the rough case which is more difficult. Firstly, by (3.5), (5.2) one can bound $y, \hat{y}$ by a generic constant $K = K(\hat{y}_a, \mathbf{x}, [a,b])$ depends on $\hat{y}_a, \mathbf{x}, [a,b]$ and consider f, D^2g Lipchitz global with constant K for convenience.

Define $h_t = y_t - \hat{y}_t$ for all $t \in \Pi[a,b]$. Then $h_a = 0$ and

$$\begin{aligned}
h_{t_k,t_{k+1}} &= [f(h_{t_k} + \hat{y}_{t_k}) - f(\hat{y}_{t_k})](t_{k+1} - t_k) + [g(h_{t_k} + \hat{y}_{t_k}) - g(\hat{y}_{t_k})]x_{t_k,t_{k+1}} \\
&\quad + [Dg(h_{t_k} + \hat{y}_{t_k})g(h_{t_k} + \hat{y}_{t_k}) - Dg(\hat{y}_{t_k})g(\hat{y}_{t_k})]\mathbb{X}_{t_k,t_{k+1}} + e_{t_k,t_{k+1}} \\
&= [f(h_{t_k} + \hat{y}_{t_k}) - f(\hat{y}_{t_k})](t_{k+1} - t_k) + F_{t_k,t_{k+1}} + e_{t_k,t_{k+1}},
\end{aligned}$$

where

- $\|e_{t_k,t_{k+1}}\| \leq \omega_{t_k,t_{k+1}}$ with control function ω of the form

$$\omega_{s,t} = K(|\Pi[a,b]| + \|\mathbf{x}\|_{p,[a,b],|\Pi[a,b]|})^{3-p} K \left[t - s + \|\mathbf{x}\|^p_{p,[s,t]}\right], \ s < t$$

in which K is a generic constant $K = K(\hat{y}_a, \mathbf{x}, [a,b])$.

- $F_{s,t} = [g(y_s)x_{s,t} + Dg(y_s)g(y_s)\mathbb{X}_{s,t}] - [g(\hat{y}_s)x_{s,t} + Dg(\hat{y}_s)g(\hat{y}_s)\mathbb{X}_{s,t}]$

Put

$$R^h_{s,t} := h_{s,t} - [g(h_s + \hat{y}_s) - g(\hat{y}_s)]\, x_{s,t}, \ (s,t) \in [a,b], s \leq t.$$

A direct computation shows that: for $s \leq u \leq t$ in $[a,b]$

$$\begin{aligned}
&\|(\delta F)_{s,u,t}\| \\
\leq\ & \Big\|\Big(g(\hat{y}_u + h_u) - g(\hat{y}_u) - g(\hat{y}_s + h_s) + g(\hat{y}_s) \\
&-[Dg(\hat{y}_s + h_s)g(\hat{y}_s + h_s) - Dg(\hat{y}_s)g(\hat{y}_s)]x_{s,u}\Big) \otimes x_{u,t}\Big\| \\
&+\|Dg(\hat{y}_u + h_u)g(\hat{y}_u + h_u) - Dg(\hat{y}_u)g(\hat{y}_u) - Dg(\hat{y}_s + h_s)g(\hat{y}_s + h_s) + Dg(\hat{y}_s)g(\hat{y}_s)\|\|\mathbb{X}_{u,t}\|.
\end{aligned}$$

Due to the assumption ($\mathbf{H_g}$), it is easy to show that $Dg(y)g(y)$ is Lipschitz continuous with the Lipschitz constant KL_g for a generic constant $K > 0$, thus the second term in the above inequality can be estimated as

$$\begin{aligned}
&\|Dg(\hat{y}_u + h_u)g(\hat{y}_u + h_u) - Dg(\hat{y}_u)g(\hat{y}_u) - Dg(\hat{y}_s + h_s)g(\hat{y}_s + h_s) + Dg(\hat{y}_s)g(\hat{y}_s)\|\|\mathbb{X}_{u,t}\| \\
\leq\ & KL_g(\|h_{s,u}\| + \|h\|_{\infty,[s,u]}\|\hat{y}_{s,u}\|)\|\mathbb{X}_{u,t}\| \\
\leq\ & KL_g\left[\|h\|_{p,\Pi[s,t]} + \|h\|_{\infty,\Pi[s,t]}(\|\hat{y}\|_{p,\Pi[s,t]} + \|y\|_{p,\Pi[s,t]})\right]\|\mathbb{X}\|_{q,\Pi[s,t]}.
\end{aligned}$$

Meanwhile, the first term has the form $\|M \otimes x_{u,t}\|$ where it can be shown that

$$\begin{aligned}
\|M \otimes x_{u,t}\| =\ & \Big\|\int_0^1 \Big(Dg(\hat{y}_s + h_s + \eta(\hat{y}_{s,u} + h_{s,u}))(\hat{y}_{s,u} + h_{s,u}) - Dg(\hat{y}_s + \eta\hat{y}_{s,u})\hat{y}_{s,u}\Big)d\eta \\
&-[Dg(\hat{y}_s + h_s)g(\hat{y}_s + h_s) - Dg(\hat{y}_s)g(\hat{y}_s)]x_{s,u}\Big\|\|x_{u,t}\| \\
\leq\ & L_g(\|R^h_{s,u}\| + \|h\|_{\infty,\Pi[s,t]}\|R^{\hat{y}}_{s,u}\|)\|x_{u,t}\| \\
&+\Big\|\int_0^1 \Big(Dg(\hat{y}_s + h_s + \eta(\hat{y}_{s,u} + h_{s,u}))g(\hat{y}_s + h_s) - Dg(\hat{y}_s + \eta\hat{y}_{s,u})g(\hat{y}_s) \\
&-Dg(\hat{y}_s + h_s)g(\hat{y}_s + h_s) + Dg(\hat{y}_s)g(\hat{y}_s)\Big)d\eta\Big\|\|x_{s,u} \otimes x_{u,t}\| \\
\leq\ & L_g(\|R^h_{s,u}\| + \|h\|_{\infty,\Pi[s,t]}\|R^{\hat{y}}_{s,u}\|)\|x_{u,t}\| \\
&+\int_0^1 \Big\{\Big\|\Big(Dg(\hat{y}_s + h_s + \eta(\hat{y}_{s,u} + h_{s,u})) - Dg(\hat{y}_s + h_s)\Big)g(\hat{y}_s + h_s) \\
&-\Big(Dg(\hat{y}_s + \eta\hat{y}_{s,u}) - Dg(\hat{y}_s)\Big)g(\hat{y}_s)\Big\|\Big\}d\eta\|x_{s,u} \otimes x_{u,t}\| \\
\leq\ & L_g(\|R^h_{s,u}\| + \|h\|_{\infty,[s,t]}\|R^{\hat{y}}_{s,u}\|)\|x_{u,t}\| + KL_g\|h_{s,u}\|\|x_{s,u} \otimes x_{u,t}\| \\
&+\int_0^1 \Big\{\Big\|\int_0^1 \Big[D^2g(\hat{y}_s + h_s + \kappa\eta y_{s,u})\eta\hat{y}_{s,u} \otimes g(\hat{y}_s + h_s) \\
&-D^2g(\hat{y}_s + \kappa\eta\{\hat{y}_{s,u})\eta\hat{y}_{s,u} \otimes g(\hat{y}_s)\Big]d\kappa\Big\|\Big\}d\eta\|x_{s,u} \otimes x_{u,t}\|
\end{aligned}$$

$$
\begin{aligned}
&\leq L_g(\|R^h_{s,u}\| + \|h\|_{\infty,\Pi[s,t]}\|R^y_{s,u}\|)\|x_{u,t}\| + KL_g\|h_{s,u}\|\|x_{s,u}\otimes x_{u,t}\| \\
&\quad + \int_0^1 \Big\{ \int_0^1 \Big\| \Big(D^2 g(\hat{y}_s + h_s + \kappa\eta\hat{y}_{s,u}) - D^2 g(\hat{y}_s + \kappa\eta\hat{y}_{s,u})\Big)\eta\hat{y}_{s,u}\otimes g(\hat{y}_s + h_s) \\
&\quad + D^2 g(\hat{y}_s + \kappa\eta\hat{y}_{s,u})\eta\hat{y}_{s,u}\otimes\Big(g(\hat{y}_s + h_s) - g(\hat{y}_s)\Big)\Big\| d\kappa\Big\} d\eta \|x_{s,u}\otimes x_{u,t}\| \\
&\leq L_g\Big(\left\|R^h\right\|_{q,\Pi[s,t]} + \|h\|_{\infty,\Pi[s,t]} \left\|R^{\hat{y}}\right\|_{q,\Pi[s,t]}\Big) \left\| x \right\|_{p,\Pi[s,t]} \| \\
&\quad + KL_g\Big(\left\| h\right\|_{p,\Pi[s,t]} + \|h\|_{\infty,\Pi[s,t]} \left\| y\right\|_{p,\Pi[s,t]}\Big) \left\|\mathbb{X}\right\|_{q,\Pi[s,t]} \\
&\leq K\Big(\left\|R^h\right\|_{q,\Pi[s,t]} \left\| x\right\|_{p,\Pi[s,t]} + \left\| h\right\|_{p,\Pi[s,t]} \left\|\mathbb{X}\right\|_{q,\Pi[s,t]}\Big) \\
&\quad + K\|h\|_{\infty,\Pi[s,t]}\Big(\left\| x\right\|_{p,\Pi[s,t]} + \left\|\mathbb{X}\right\|_{q,\Pi[s,t]}\Big).
\end{aligned}
$$

As a result, one can apply Corollary 2.4 to conclude that

$$
\begin{aligned}
\|h_{s,t}\| &\leq K\|h\|_{\infty,\Pi[s,t]}(t-s) + K\|h\|_{\infty,\Pi[s,t]}(\|x_{s,t}\| + \|\mathbb{X}_{s,t}\|) + \omega_{s,t} \\
&\quad + KL_g\Big(\left\|R^h\right\|_{q,\Pi[s,t]} \left\| x\right\|_{p,\Pi[s,t]} + \left\| h\right\|_{p,\Pi[s,t]} \left\|\mathbb{X}\right\|_{q,\Pi[s,t]}\Big); \\
\|R^h_{s,t}\| &\leq C_f\|h\|_{\infty,\Pi[s,t]}(t-s) + KL_g\|h\|_{\infty,\Pi[s,t]}\|\mathbb{X}_{s,t}\| + \omega_{s,t} \\
&\quad + K\Big(\left\|R^h\right\|_{q,\Pi[s,t]} \left\| x\right\|_{p,\Pi[s,t]} + \left\| h\right\|_{p,\Pi[s,t]} \left\|\mathbb{X}\right\|_{q,\Pi[s,t]}\Big).
\end{aligned}
$$

One can now apply the same arguments as in the proof of Theorem 3.3 to show that there exists a random variable $\hat{\xi}(\hat{y}_a, \mathbf{x}, [a,b])$ such that

$$
\begin{aligned}
\|h\|_{\infty,[a,b]} &\leq \|(h,R^h)\|_{p,[a,b]} \leq (\|h_a\| + \omega_{a,b})\hat{\xi}(\hat{y}_a, \mathbf{x}, [a,b]) \\
&\leq K(|\Pi[a,b]| + \left\|\mathbf{x}\right\|_{p,[a,b],|\Pi[a,b]|})^{3-p}
\end{aligned}
$$

which proves (5.3) for $\hat{\xi}$ generic. Finally, because $p \in [2,3)$ and $\mathbf{x} \in C^{\frac{1}{p}}$ is of $\frac{1}{p}$- Hölder rough path, it is easy to check that $\left\|\mathbf{x}\right\|_{p,[a,b],|\Pi[a,b]|} \to 0$ as $|\Pi[a,b]| \to 0$. This proves (5.4).

□

Next we recall that in [10], it is proved that if f satisfies dissipative condition, (5.1) generate a RDS φ which possesses a pullback random attractor $\mathcal{A}$. At the same time (5.1) generate a discrete RDS which possesses a random pullback attractor $\mathcal{A}^\Delta$ when $\Delta \leq \Delta_0$ small enough. In the following we prove that the discrete attractor converges to the continuous attractor as Δ approaches 0. Note that the result here is obtained without the condition on boundedness of f as proved in [10].

To satisfy condition (4.8) for any step size Δ, we assume a stronger condition that

$$
\mathbb{E}\left\|\mathbf{x}\right\|^k_{p,[a,b]} < \infty, \quad \forall k \in \mathbb{N}, \quad \forall 0 < a < b. \tag{5.5}
$$

We first prove that

Proposition 5.2 *Under assumptions* ($\mathbf{H_f}$)*,* ($\mathbf{H_g}$)*,* (4.7) *and* (5.5)*, there exists $L > 0$ such that if $L_g < L$, for all Δ small enough the attractor $\mathcal{A}^\Delta$ of* (1.2) *is uniformly bounded in Δ a.s.*

Proof: For T and $R^\Delta(\mathbf{x})$ in Theorem 4.2 where Δ small enough so that $T > 1/2$, observe that for each k, $n_k = \lfloor kT \rfloor$ satisfies $n_k \leq kT < (k+1)T < n_k + 2$ and $n_k \leq n_{k+1} \leq n_k + 1, n_k < n_{k+2}$. We then have some generic constant M (note that $n_k \leq k$)

$$
R^\Delta(\mathbf{x}) \leq 1 + M\sum_{k=0}^\infty e^{-dk}\Lambda(\mathbf{x}, [-k, -k+2])] \prod_{j=0}^k \Big[1 + L_g\Lambda(\mathbf{x}, [-j-1, -j+1])\Big]^2
$$

$$\leq \quad 1 + M \sum_{k=1}^{\infty} e^{-dk} \Lambda(\theta_{-k}\mathbf{x}, [0,2]) \prod_{j=1}^{k} \Big[1 + L_g \Lambda(\theta_{-j}\mathbf{x}, [0,2])\Big]^2$$

$$\leq \quad 1 + M \sum_{k=1}^{\infty} e^{-dk} \Lambda(\theta_{-k}x, [-1,2]) \prod_{j=1}^{k} \Big[1 + L_g \Lambda(\theta_{-j}\mathbf{x}, [-1,2])\Big]^2 =: R(\mathbf{x}). \tag{5.6}$$

Hence, using similar arguments in Theorem 4.2 one can choose L_g small enough and independent of Δ such that the series in (5.6) converges almost surely. That implies $\mathcal{A}^\Delta \subset \bar{B}(0, R^\Delta(\mathbf{x})) \subset \bar{B}(0, R(\mathbf{x}))$ for all Δ satisfying (4.9) where the radius $R(x)$ is independent of Δ. The proof is finished. □

Theorem 5.3 (Convergence of numerical attractor) *Under conditions in Proposition 5.1 and (5.5), for L_g small enough, the numeric attractor $\mathcal{A}^\Delta$ converges to the attractor $\mathcal{A}$ in the Hausdorf semi-distance, i.e. $d_H(\mathcal{A}^\Delta|\mathcal{A}) \to 0$ as $\Delta \to 0$, a.s.*

Proof: First, as indicated in (5.6) of Proposition 5.2, for L_g small enough so that $R(\mathbf{x})$ is finite, we have $R^\Delta(\mathbf{x}) \leq R(\mathbf{x})$ for all $\Delta < \Delta_0$ small. On the other hand, assumption ($\mathbf{H_f}$) and the dissipativity condition (4.7) match the conditions in [7, Theorem 3.1] for g bounded and [10, Theorem 5.1] for g linear to conclude the existence of a random attractor $\mathcal{A}$.

Now, we proceed a contradiction arguments. Namely, assume the assertion is false for a particular ω corresponding to a rough path $\mathbf{x}$, then there exists an $\varepsilon_0 > 0$ and a sequence $\Delta_j \downarrow 0^+$ such that $d_H(\mathcal{A}^{\Delta_j}(\omega)|\mathcal{A}(\omega)) > \varepsilon_0$ for all $j \in \mathbb{N}$. Since these attractors are compact sets, there exists $a_j(\omega) \in \mathcal{A}^{\Delta_j}(\omega)$ such that $d_H(a_j(\omega)|\mathcal{A}(\omega)) > \varepsilon_0$. Due to the invariance, there exists for each $m_j \in \mathbb{N}$ a point $b_j(\theta_{-m_j\Delta_j}\omega) \in \mathcal{A}^{\Delta_j}(\theta_{-m_j\Delta_j}\omega) \subset \mathcal{B}^{\Delta_j}(\theta_{-m_j\Delta_j}\omega)$ such that $\varphi^{\Delta_j}(m_j\Delta_j, \theta_{-m_j\Delta_j}\omega)b_j(\theta_{-m_j\Delta_j}\omega) = y^{\Delta_j}_{m_j\Delta_j}(\theta_{-m_j\Delta_j}\omega, b_j(\theta_{-m_j\Delta_j}\omega)) = a_j(\omega)$. Respectively one considers the continuous flow with respect to the rough differential equation

$$\varphi(m_j\Delta_j, \theta_{-m_j\Delta_j}\omega, b_j(\theta_{-m_j\Delta_j}\omega)) = \hat{y}_{m_j\Delta_j}(\theta_{-m_j\Delta_j}\omega, b_j(\theta_{-m_j\Delta_j}\omega))$$

and applies the triangle inequality to obtain

$$\begin{aligned}\varepsilon_0 < d_H(a_j(\omega)|\mathcal{A}(\omega)) \quad &\leq \quad \|\hat{y}_{m_j\Delta_j}(\theta_{-m_j\Delta_j}\omega, b_j(\theta_{-m_j\Delta_j}\omega)) - y^{\Delta_j}_{m_j\Delta_j}(\theta_{-m_j\Delta_j}\omega, b_j(\theta_{-m_j\Delta_j}\omega))\| \\ &\quad + d_H(\varphi(m_j\Delta_j, \theta_{-m_j\Delta_j}\omega, b_j(\theta_{-m_j\Delta_j}\omega))|\mathcal{A}(\omega)).\end{aligned} \tag{5.7}$$

On the other hand, since $\mathcal{A}$ is the pullback attractor of φ, it attracts also $\bar{B}(0, R(\omega))$ in the pullback sense, thus there exists a fixed $T^* := T(\varepsilon_0, \omega)$ such that for any $m_j\Delta_j \in [T^*, T^*+1]$

$$d_H(\varphi(m_j\Delta_j, \theta_{-m_j\Delta_j}\omega, b_j(\theta_{-m_j\Delta_j}\omega))|\mathcal{A}(\omega)) \leq \frac{\varepsilon_0}{2}. \tag{5.8}$$

Meanwhile, due to the estimate (5.3) in Proposition 5.1, the first difference in the right hand side of (5.7), for a fixed interval $[0, T^*+1]$, is uniform with respect to the initial point $b_j(\theta_{-m_j\Delta_j}\omega)$ in the compact set $\mathcal{B}^{\Delta_j}(\theta_{-m_j\Delta_j}\omega)$. Hence one obtains for Δ_j small enough

$$\begin{aligned}&\|\hat{y}_{m_j\Delta_j}(\theta_{-m_j\Delta_j}\omega, b_j(\theta_{-m_j\Delta_j}\omega)) - y^{\Delta_j}_{m_j\Delta_j}(\theta_{-m_j\Delta_j}\omega, b_j(\theta_{-m_j\Delta_j}\omega))\| \\ \leq \quad & K(R^{\Delta_j}(\theta_{-m_j\Delta_j}\omega)) \Big[\Delta_j + \vert\!\vert\!\vert \omega \vert\!\vert\!\vert_{p,[-T^*-1,1],\Delta_j}\Big]^{3-p} \Big[T^* + 1 + \vert\!\vert\!\vert \omega \vert\!\vert\!\vert^p_{p,[-T^*-1,1]}\Big].\end{aligned} \tag{5.9}$$

Write $m_j\Delta_j = T^* + s_j$ with $s_j \in [0,1]$, one has

$$R^{\Delta_j}(\theta_{-m_j\Delta_j}\omega)$$

$$
\begin{aligned}
&\leq \quad 1 + M \sup_{s\in[0,1]} \sum_{k=0}^{\infty} e^{-dk} \Lambda(\theta_{-k}\theta_{-T^*}\omega, [0-s, 2-s]) \prod_{j=0}^{k} \Big[1 + L_g \Lambda(\theta_{-j}\theta_{-T^*}\mathbf{x}, [-s, 2-s])\Big]^2 \\
&\leq \quad 1 + M \sum_{k=0}^{\infty} e^{-dk} \Lambda(\theta_{-k}\theta_{-T^*}x, [-1, 2]) \prod_{j=0}^{k} \Big[1 + L_g \Lambda(\theta_{-j}\theta_{-T^*}\omega, [-1, 2])\Big]^2 \\
&\leq \quad R(\theta_{-T^*}(\omega)). \qquad (5.10)
\end{aligned}
$$

Hence, the right hand side of (5.9) tends to 0 as $j \to \infty$, i.e.

$$
\|\hat{y}_{m_j\Delta_j}(\theta_{-m_j\Delta_j}\omega, b_j(\theta_{-m_j\Delta_j}\omega)) - y^{\Delta_j}_{m_j\Delta_j}(\omega, b_j(\theta_{-m_j\Delta_j}\omega))\| \leq \frac{\varepsilon_0}{2} \qquad (5.11)
$$

for j large enough. Since (5.8) and (5.11) contradict to (5.7), this completes the proof. □

Remark 5.4 In Theorem 4.2 and 5.3, the results are established for the discrete system under condition "sufficiently small L_g", which implies that g is not necessarily bounded but its derivatives must be small enough. Therefore in case g is bounded, this condition can be relaxed, but we need a little stronger condition than (5.5) that

$$
\mathbb{E}\, \|\mathbf{x}\|^k_{1/p\text{-Hol},[a,b]} < \infty, \quad \forall k \in \mathbb{N}, \quad \forall 0 < a < b, \qquad (5.12)
$$

In the following, we would like to clarify the argument for discrete system (1.2). Indeed, in Theorems 3.3 and 4.2 we construct the sequence $\tau_m(\delta, [0,T])$ on $[0,T]$ and obtain the estimate

$$
\sum_{m=0}^{N(\delta,[0,T])-1} \|y, R^y\|_{p,\Pi[\tau_m,\tau_{m+1}]} \leq K \Big[T\|y\|_{\infty,\Pi[0,T]} + \xi(\|\mathbf{x}\|_{p,\Pi[0,T]})\Big]
$$

for a sufficiently small δ, a generic polynomial ξ and a generic constant K. We modify (4.11) as

$$
\|y_{t_n}\|^2 \leq e^{-dT}\|y_0\|^2 + K\left(1 + \|f(y)\|_{\infty,\Pi[0,T]}\|F\|_{\infty,\Pi[0,T]}\right) + \sum_{i=0}^{N(\delta,[0,T])-1} \Big[G_{\tau_i,\tau_{i+1}} + K\omega^{\lambda_{\tilde{G}}}_{\tau_i,\tau_{i+1}}(\delta\tilde{G})\Big].
$$

Since g is bounded, we use the estimate $\|F\|_{\infty,\Pi[s,t]} \leq L_g \|\mathbf{x}\|_{p,\Pi[s,t]}$ and (3.3), and choose $\delta = (1 \wedge d(1-\Delta_0))/2K$. Then a direct calculation yields

$$
\begin{aligned}
\|y_T\|^2 \quad &\leq \quad e^{-dT}(1 + K\delta + L_g\Delta^{1/p}\|\mathbf{x}\|_{1/p-\text{Hol},\Pi[0,T]})\|y_0\|^2 + Ke^{\xi(\|\mathbf{x}\|_{1/p-\text{Hol},\Pi[0,T]})} \\
&\leq \quad e^{-dT/2}(1 + L_g\Delta^{1/p}\|\mathbf{x}\|_{1/p-\text{Hol},\Pi[0,T]})\|y_0\|^2 + Ke^{\xi(\|\mathbf{x}\|_{1/p-\text{Hol},\Pi[0,T]})},
\end{aligned}
$$

which is similar to (4.14). The role of $L_g\Delta^{1/p}$ is similar to L_g as a coefficient of $\|y_0\|^2$ in (4.14). Therefore, Theorem 4.2 is prove for g bounded with L_g be arbitrary. This coincides to the results present in [10] for rough cases and in [11] for Young cases.

As a sequence of Theorem 4.2, the convergence in Theorem 5.3 holds for arbitrary L_g. This is an improvement of [10] where both f and g need to be bounded.

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Declarations

- **Ethical Approval** (applicable for both human and/ or animal studies. Ethical committees, Internal Review Boards and guidelines followed must be named. When applicable, additional headings with statements on consent to participate and consent to publish are also required): not applicable
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